SOME SURREAL NUMBERS ARGUMENTS

Unlike previous sheets, the *Surreal Numbers* sheet is intended to be much more open-ended. I don't necessarily want to see proof of the results listed on the sheet, but the questions are there to spark your imagination and guide you to formulate your own questions that you might try and answer. The ideal way to answer the later questions is to formulate your own claim and prove it, which ultimately answers the question.

Question 2. What are all the different ways to express each number in S_2 with elements of S_1 allowed for L and R? (We haven't really defined what it means for an arbitrary surreal number $\{L|R\}$ to have a value, so we can't really prove this yet, but this first question is just to build some intuition.)

Solution. The equalities of values are

$$\begin{aligned}
-2 &= \{|-1\} = \{|-1,0\} = \{|-1,1\} = \{|-1,0,1\} \\
-1 &= \{|0\} = \{|0,1\} \\
-\frac{1}{2} &= \{-1|0\} = \{-1|0,1\} \\
0 &= \{|\} = \{-1|\} = \{|1\} = \{-1|1\} \\
\frac{1}{2} &= \{0,1\} = \{-1,0|1\} \\
1 &= \{0|\} = \{-1,0|\} \\
2 &= \{1|\} = \{0,1|\} = \{-1,1|\} = \{-1,0,1|\}
\end{aligned}$$

Note that we have the obvious symmetry under negation, and this should also give you an idea for the answer to Question 6 (but perhaps not how to prove it). □

Again, to prove these equalities we need to use the definition of $\geq$. But it's tedious to just try and use the definition of $\geq$ all the time and an initial difficulty in learning the subject is how to determine the value of a number like $\{-1,1|\}$. How you want to proceed is to recognize patterns (in both examples and arguments) and summarize them into claims that you can prove and then apply. Here are examples of follow-up questions you can try and answer (with tools like $\leq$) that you may find useful. Are they true or not?

- If y is a number that is smaller than the greatest element of L, then $\{y, L|R\} = \{L|R\}$?
- If y is a number is greater than the smallest element of R, then $\{L|R, y\} = \{L|R\}$?
- For any positive integer n, must we $n = \{0, 1, 2, \ldots, n-1|\}$?
- If a number is born on day n, must it be the mean of the greatest element of L and the smallest element of R?

Of course, this is just the tip of the iceberg, and you may have an entirely different way to thinking about the topic, which would be great if you do!

As almost all of the definitions of surreal numbers involve induction, proving a general result should involve induction as well! Here is an example of how one such argument.

Proposition 4. Show that $x \geq x$ for all $x \in \mathbf{No}$.

Proof. We prove this by induction on days.

Base case: On day 0, we only have 0, so we want to show that $0 \geq 0$. Since $0 = \{|\}$ and so has both empty L and R, both conditions required for $0 \geq 0$ are vacuously true. Thus $0 \geq 0$.

Inductive step: Suppose that $x \geq x$ is true for all numbers constructed up to day n. We want to show that if x has a birthday on day $n+1$, then $x \geq x$. Since x is born on day $n+1$, we can write $x = \{L|R\}$ where all numbers in L and R have birthday on day n or earlier.

Suppose for the sake of contradiction that $x \geq x$ is not true. We have two ways in which this holds: either (a) there exists an $x^R \leq x$ or (b) there exists an $x^L \geq x$. We will show that both cases lead to a contradiction.

Case (a): Suppose that we have an element $x^R \in R$ such that $x^R \leq x$. Then there must not exist any element $(x^R)' \in R$ such that $(x^R)' \leq (x^R)$. However, x^R itself is in R and has the property that $x^R \leq x^R$ by the inductive hypothesis (as it was created on day n or earlier), so we obtain a contradiction.

Case (b): Suppose that there existsn an element $x^L \in L$ such that $x^L \geq x$. Then there must exist any element $(x^L)' \in L$ such that $(x^L)' \geq x^L$. However, x^L itself is in L and has the property that $x^L \geq x^L$ by the inductive hypothesis, so we obtain a contradiction.

Thus, we conclude that $x \geq x$ for any number born on day $n+1$, as desired. □

There's lots of room for cleverness here. While induction is often the most straightforward way, you can sometimes find ways to avoid it and find a cool proof. Let's talk about answering Question 4 as an example case.

Related Question 4. Given $x = \{L|R\}$, we define

$$-x = \{-R| - L\}$$

where $-S = \{-x \mid x \in L\}$ for any set of numbers S. (Note that if x is created on day n, then $-x$ is also created on day n. Prove it, if it's not clear!)

We want to show that $x + (-x) = 0$. There are a number of ways to do so (e.g. using Question 8.) But here is one way to do it "with your bare hands."

Proof. From the definition of addition, we have

$$x+(-x) = \{x+(-L), (-x)+L|x+(-R), (-x)+R\} = \{x+(-x)^L, (-x)+x^L|x+(-x)^R, (-x)+x^R\}.$$

How can we show a number equals zero? There are a number of ways. For one, note that zero is the only number such that $-y = y$. We have

$$\begin{aligned}
-\{x+(-x)^L, (-x)+x^L|x+(-x)^R, (-x)+x^R\} &= -\{x+(-R), (-x)+L|x+(-L), (-x)+R\} \\
&= \{-(x+(-L)), -((-x)+(R))| - (x+(-R), -((-x)+L)\} \\
&= \{-x+L, x+(-R)| - x+R, x+(-L)\} \\
&= \{-x+x^L, x+(-x)^L| - x+x^R, x+(-x)^R\}
\end{aligned}$$

as desired. □