RELATIONS

Definition 1. Let A and B be two sets. A **(binary) relation on** A **and** B is a subset $R \subseteq A \times B$. If $a \in A$ and $b \in B$ are such that $(a, b) \in R$, then we say that a **is (R-)related to** b (written as $a \sim_R b$ or simply $a \sim b$ when it it clear what the relation is). If $A = B$, we often just say that R is a (binary) **relation on** A.

Examples 2. Consider the set $S = \{1, 2, 3\}$. Since $S \times S$ has nine elements, $\mathcal{P}(S)$ has $2^9 = 512$ elements, so there are 512 binary relations on S and S. Familiar examples include equality $R_= = \{(1, 1), (2, 2), (3, 3)\}$, ordering by size $R_< = \{(1, 2), (1, 3), (2, 3)\}$, and divisibility $R_| = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}$.

Question 3. *Why is a function a certain kind of relation? How can you characterize which relations are functions?*

Most of these relations aren't so interesting, so we focus our study on ones with certain properties.

Definition 4. A relation R on a set A is said to be

- **reflexive** if $a \sim a$ for all $a \in A$;
- **symmetric** if $a, b \in A$ and $a \sim b$, then $b \sim a$;
- **transitive** if $a, b, c \in A$ are such that $a \sim b$ and $b \sim c$, then $a \sim c$.

Example 5. The relation $<$ on $\mathbb{R}$ is not reflexive, not symmetric, but is transitive. The relation of divisibility in $\mathbb{N}$ is reflexive, *anti*-symmetric (i.e. $a \not\sim b$ *implies* $b \sim a$ and vice versa), and transitive.

Question 6. *Let P be the set of people at a party and define a relation R on P, where $(x, y) \in R$ if and only if x knows the name of y. Describe what it would mean for R to be reflexive, symmetric, or transitive.*

Question 7. *If a relation is symmetric and transitive, is it necessarily reflexive?*

Definition 8. A relation R on a set A is an **equivalence relation** if R is reflexive, symmetric, and transitive. If $a, b \in A$ such that $a \sim b$, we say that a and b lie in the same **equivalence class**. We write $[a]$ for the equivalence class containing a (with respect to the relation R).

Example 9. What's an example of an equivalence relation that is not equality?

Question 10. *Let A be a set of 4 elements. How many relations are there on A? Reflexive relations? Symmetric relations? Functions from A to A? Equivalence relations?*

Definition 11. A **partition** of a nonempty set A is a subset S of $\mathcal{P}(A)$ such that

(1) $\emptyset \notin S$;
(2) If $x, y \in S$, then either $x = y$ or $x \cap y = \emptyset$; and
(3) The union of all elements in S equals A.

Equivalently, it a decomposition of A into a disjoint union of nonempty subsets.

Theorem 12. *(Fundamental Theorem of Equivalence Relations) Let A be a nonempty set. If R is an equivalence relation on A, then the equivalence classes (with respect to R) form a partition of A. Conversely, if S is a partition of A and we define a relation R on A such that $a \sim b$ if and only a and b are in the same set in the partition, then R is an equivalence relation on A.*

Question 13. *Consider the relation on the set* $[6]$ *given by*

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5), (6, 6), (5, 6), (6, 5), (4, 6), (6, 4).\}$$

Check that it is an equivalence relation. What is the partition corresponding to R?