MATH 3040: HOMEWORK 4

All answers must be in complete sentences. Remember that this is a writing class, so while correctness and thoroughness are most important, *style* matters as well.

For some problems, it may be useful to use a variant of induction called **strong induction**. (See [TAP, §4.5] for details.)

1. (Practice in Posing Questions) What is an interesting open-ended question you could ask, which is related to the material that we covered in class? What is a concrete mathematical question that could be used to approach an answer to the question?

2. (Fibonacci Numbers) The Fibonacci sequence F_i is a sequence of numbers defined recursively by $F_1 = 1$, $F_2 = 2$, and $F_{n+2} = F_n + F_{n+1}$ for $n \geq 1$.

(a) Show by induction on n that

$$F_n < \left(\frac{1+\sqrt{5}}{2}\right)^n.$$

(b) Prove that F_n is the closest integer to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}.$$

(This allows us to compute F_n even faster than the formula in Prop. 4.29 of [TAP].)

(c) For all $n \in \mathbf{N}$, we have $F_{2n+2} = F_n^2 + F_{n+1}^2$.

Remark. The number $\varphi = \frac{1+\sqrt{5}}{2}$ is known as the *golden ratio*, and shows up in a number of natural processes and in many things that we as humans consider to be aesthetically pleasing.

3. (Finding Formulas) Suppose that $n \in \mathbf{N}$. Find and prove a closed formula (e.g. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$) for each of the following expressions.

(a) $1 + 2 + 4 + 8 + \cdots + 2^n$.

(b) $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!$ (Recall that factorials are defined as, e.g. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.)

(c) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)}$

(d) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$.

Date: September 13, 2017.

(e) $(1 - \frac{1}{4}) \cdot (1 - \frac{1}{9}) \cdots (1 - \frac{1}{(n+1)^2})$.

4. (Discover the truth, and prove it!) For each of the following, determine (with proof) the set of *all* natural numbers n for which the statement holds.

(a) $2^n < n!$
(b) $2^n > n^2$
(c) $3^n + 5$ is divisible by 8. (Hint: $3^{k+2} + 5 = (3^k + 5) + 8 \cdot 3^k$ for every $k \in \mathbf{N}$; prove this fact if you want to use it.)
(d) $3^n + 4^n$ is divisible by 13.

5. (Induction and Beyond)

(a) Define an infinite sequence $(a_1, a_2, a_3, \ldots)$ recursively by $a_1 = 2$, $a_{n+1} = a_n + 2n$ for all $n \in \mathbf{N}$. Find and prove an explicit formula for a_n (i.e. a formula in terms of n only).
(b) Find a formula for the number of regions created by n pairwise intersecting circles in the plane in general position (i.e. every pair of circles intersects in two points and no three of the circles intersect at the same point).

The following problem should be submitted as a group. Modify the groupwork template tex file accordingly. You can look at the .tex file for this homework assignment if you don't want to retype it in.

6. (Groupwork) [Survivors among the natural numbers] Given positive integers $x_1, x_2, \ldots, x_n$, let $A(x_1, \ldots, x_n)$ be the set of natural numbers that can be written as a sum of (positive) multiples of the x_i's, that is, in other words, every number $n \in \mathbf{N}$ of the form

$$n = a_1 \cdot x_1 + a_2 \cdot x_2 + \cdots + a_n \cdot x_n$$

for any $a_1, \ldots, a_n \in \mathbf{N}$ is contained in $A(x_1, \ldots, x_n)$. Define the **last survivor number** $S(x_1, \ldots, x_n)$ of $x_1, x_2, \ldots, x_n$ to be the maximum element of $\mathbf{N}$ that is *not* contained in $A(x_1, \ldots, x_n)$, if it exists.

(a) Explain why $S(x_1, x_2, \ldots, x_n)$ can only exist if $x_1, x_2, \ldots, x_n$ are relatively prime (i.e. no two numbers share a common factor). (*Fun fact.* The converse statement is true: if the integers are relatively prime, then the last survivor number exists.)
(b) Prove that $S(4, 5) = 11$.
(c) Find $S(3, 10)$ and prove your answer.
(d) Find $S(3, 5, 7)$ and prove your answer.
(e) Find $S(7, 19, 37)$ and prove your answer.
(f) Assume that a and b are relatively prime. Formulate a conjecture for what $S(a, b)$ should be, in terms of a and b. (You do not need to prove your claim, but you should explain how you formulated it.)

(g) Assume that a and b are relatively prime. Formulate a conjecture for the number of **survivors** (i.e. the natural numbers that do not lie in $S(a, b)$), in terms of a and b. (You do not need to prove your claim, but you should explain how you formulated it.)