MATH 3040: HOMEWORK 9

All answers must be in complete sentences. All assertions must be proven, per the student contract. Remember that this is a writing class, so while correctness and thoroughness are most important, *style* matters as well.

1. (Practice in Posing Questions) What is an interesting open-ended question you could ask, which is related to the material that we covered in class? What is a concrete mathematical question that could be used to approach an answer to the question?

2. (Wrap-up)

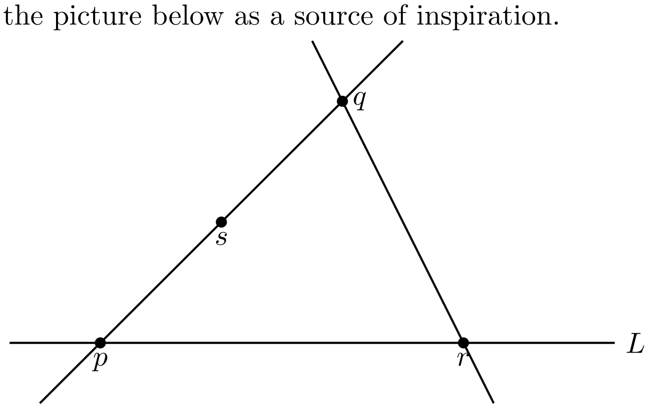
(a) Show that there are at least three points on any line in a projective plane.

(b) Give an example of a structure that satisfies axioms (P1) and (P2), where there are two lines with a different number of points that lie on them.

3. (Wrap-up) Let Π be a finite projective plane. We want to prove that the number of lines through any point is the same as the number of points on any line. We will walk through a proof of this.

(a) Let L be any line and p any point that is not on L. Construct a bijection between the set of lines through p and the set of points on L. Make sure you prove that this map is well-defined, injective, and surjective.

(b) Extend the result to the case where the point p is on the line L. You can use the picture below as a source of inspiration.



(c) Now prove, with little effort, that

- the same number of lines pass through every point of Π;

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- the same number of points lie on every line of Π.

4. (Points in a Projective Plane) Let Π be a finite projective plane of order q.

(a) Show that the number of points in Π is equal to the number of lines in Π. (Hint: Find two different ways to count pairs (p, L) of points and lines where $p \in L$.)
(b) How many points are in Π? (Hint: Fix a point, count the lines through that point, and count the number of points on each of these lines. Did you count all the points? Did you count any of them more than once?)
(c) How many points are on an affine plane whose "projective completion" (See Prop. 8) is a projectve plane of order q?

5. (Countability)

(a) Every countable union (i.e. a union of countably many sets) of countable sets is countable. (Hint: there are three cases to consider finite unions of countable sets [see Prop. 9 on *Infinities*], countably infinite unions of finite sets, and countably infinite unions of countably infinite sets.)
(b) If A and B are countable sets, then $A \times B$ is countable.
(c) The set of all finite sequences of 0's and 1's (e.g. 011000110 is one such sequence) is countable.

The following question is to be done as a group.

6. (Groupwork) ("WLOG" part 2)

(a) Consider a variant of the same game as last week (connecting four points on a chalkboard, no three of which lie on the same straight line), but now the two players use different colored chalk to draw line segments: player A uses red and player B uses blue. They are allowed to connect any two of the four points that have not been connected yet (same as before), but in this game they are not allowed to form a single-colored triangle (i.e. whose three sides have the same color). The first player who is forced to create a single-colored triangle loses. If the game ends without either player creating such a triangle, we say that it is draw. Say that A moves first, as before. does either player have a winning strategy? Must each game end in a tie?
(b) What about if look at the colored variant with have five points, no three of which are collinear. Is it possible, even assuming non-optimal play on the part of A and/or B, that the two players end the game in a tie? (Hint: DO NOT try and attempt to analyze who has the winning strategy here, it is extremely intricate!)
(c) Is a tie possible in the colored variation of the game with six points?

7. (Groupwork) (Prove or Disprove and Salvage)

(a) In every group of five people, there are two people who know the same number of people in the group. (Assume throughout that "knowing someone" is mutual; if you know them, they they know you.)

(b) In every group of five people, at least one of the following groups exist: a group of three "friends" who all know each other, or a group of three "total strangers" in which no two members of this group know each other.

(c) In every group of six people, at least one of the following groups exist: a group of three "friends" or a group of three "total strangers."