MATH 3110: Homework #10
(due Tuesday, November 8)

November 1, 2016

The references to [**Mattuck**] refer to the **Exercises** at the end of chapter. If you'd like to receive full credit, be sure to show your reasoning. Try to write in complete sentences.

1. [**Mattuck 16.2/2, Problem 16-2**] (16.2/2) Assume that $f(x)$ is differentiable on I; prove that if $f(x)$ is geometrically convex, then it is convex.

(Problem 16-2) Prove that if $f(x)$ is differentiable on I and convex, then it is geometrically convex.

2. [**Mattuck 18.2/1, 2, 3**] (18.2/1) Prove directly from the definition of integrability that x^2 is integrable on any $[a, b] \subset [0, \infty)$.

(18.2/2) Let

$$f(x) = \begin{cases} 0, & \text{if there exist integers } m, n > 0 \text{ such that } x = m/2^n \\ 1, & \text{otherwise.} \end{cases}$$

Prove $f(x)$ is not Riemann-integrable on $[0, 1]$.

(18.2/3) Suppose $f(x)$ is integrable on $[a, b]$. Prove it is also integrable on any subinterval $[c, d] \subset [a, b]$.

3. [**Mattuck 18.3/2, 3**] (18.3/2) Prove directly from the definition of integrability: on an interval $[a, b]$, if $f(x)$ is differentiable and $f'(x)$ is bounded, then $f(x)$ is integrable.

(Of course, this follows from Theorem 18.3B, since a differentiable function is continuous; but prove it directly from the definition of integrability, without quoting 18.3B or any exercises on uniform continuity.)

(18.3/3) Prove that the function $f(x) = \sin(1/x), x \neq 0;\ f(0) = 0$ is integrable on $[-1, 1]$. (You may use the theorems of this section.)

4. [**Mattuck 18.3/1, 19.2/4, 19.3/1**] (18.3/1) Let n be a fixed integer > 0; consider the function

$$f_n(x) = \begin{cases} 0, & \text{if } x = 1/n, 2/n, \ldots, (n-1)/n; \\ 1, & \text{otherwise.} \end{cases}$$

Prove directly from the definition of integrability that $f_n(x)$ is integrable on $[0, 1]$.

(19.2/4) According to Exercise 18.3/1, the function $f_n(x)$ is integrable on $[0, 1]$. Assuming this, evaluate $\int_0^1 f_n(x)\,dx$, using only the results in this section.

(19.3/1) Suppose a function $f(x)$ is integrable on $[a, b]$ and $f(x) = 0$ whenever x is rational. (You are not told anything about the value of $f(x)$ when x is irrational.) Prove that $\int_a^b f(x)\,dx = 0$.

5. [**Mattuck Problem 19-1**] (a) Assume $f(x)$ integrable on I and $a, x \in I$. Prove $F(x) = \int_a^x f(t)\,dt$ is continuous on I. (Use several theorems of this chapter.)

(b) Suppose I is compact and $f(x)$ is integrable on I. Define

$$G(x, y) = \int_x^y f(t)\,dt.$$

Prove there exist points x_0 and y_0 such that $G(x_0, y_0) = \max_{x,y \in I} G(x, y)$.

(Don't try to use guessed-at theorems about functions of two variables.)

6. [**Mattuck Problem 19-2**] Here is the definition of integrable and integral that is similar to the one have been using in class. (With a little work, one can show that it's equivalent, but I won't ask you to do this.) This problem asks you to prove that it is equivalent to the formal definition in your book.

> **Sequential Definition.** Let $f(x)$ be bounded on $[a, b]$, and $\mathcal{I}$ a number. If for every sequence $\mathcal{P}_k$ of partitions of $[a, b]$ such that $|\mathcal{P}_k| \to 0$, and for every Riemann sum $S_f(\mathcal{P}_k)$ for $f(x)$ over the partition $\mathcal{P}_k$, we have
>
> $$\lim_{k \to \infty} S_f(\mathcal{P}_k) = \mathcal{I},$$
>
> then we say that $f(x)$ is integrable on $[a, b]$ and write $\mathcal{I} = \int_a^b f(x)dx$.

(a) Prove that if $f(x)$ is integrable on $[a, b]$, and $\mathcal{I}$ is its integral, in the sense of Definitions 18.2 and 19.2, then it is integrable and $\mathcal{I}$ is its integral also in the sense of the Sequential Definition above. (This is easy, using theorems.)

(b) Prove the converse: if $f(x)$ on $[a, b]$ is integrable with integral $\mathcal{I}$ in the sense of the above definition, then it is integrable and its integral is $\mathcal{I}$ in the sense of definitions 18.2 and 19.2. (Not quite so easy.)