MATH 3110: Homework #11 (due Tuesday, November 22)

November 15, 2016

The references to [**Mattuck**] refer to the **Exercises** at the end of chapter. If you'd like to receive full credit, be sure to show your reasoning. Try to write in complete sentences.

1. [**Mattuck 16.1/1, 3**] (16.1/1) Using Theorem 16.1B, prove these estimates, on the given interval.

(a) $e^x > 1 + x + x^2/2$, for $x \in [0, 1]$; find a quadratic upper estimate;

(b) $\ln(1 + x) > x - x^2$, for $x \in [0, 1]$; find a quadratic upper estimate.

(16.1/3) In the neighborhood of what point $a \in [0, 2]$ will the linearization at a of $3x + 4x^3 - x^4$ give the least accurate approximation of the function; why?

2. [**Mattuck Problem 16-3**] **The error in linear interpolation.** To estimate $f(1.13)$, knowing $f(1.1)$ and $f(1.2)$, *linear interpolation* is used: you assume $f(x)$ is linear between 1.1 and 1.2, and calculate the value of $f(1.13)$ accordingly. The problem is to estimate the error in this procedure.

So suppose $f''(x)$ exists on $[a, b]$, and let $L(x)$ be the linear function agreeing with $f(x)$ at the endpoints. Then

$$e(x) = f(x) - L(x)$$

measures the error in the approximation $f(x) \approx L(x)$.

What is the maximum value of $|e(x)|$ on $[a, b]$? Estimate it as follows.

1. Let X_0 be a maximum point for $|e(x)|$; say x_0 lies in the right half of the interval $[a, b]$. We may assume that x_0 is a *local* maximum or minimum point for $e(x)$; justify this.

2. Use Theorem 16.1B to express $e(b)$ in terms of $e(x_0)$, $e'(x_0)$, and powers of $(b - x_0)$. Deduce from this that

$$|e(x_0)| \leq \frac{|e''(c)|}{8}(b-a)^2, \quad \text{where } a < c < b. \qquad (*)$$

3. Deduce from part (b) that if $|f''(x)| \leq M$ on $[a, b]$, we get the estimate

$$|e(x)| \leq \frac{M}{8}(b-a)^2, \quad \text{for } a \leq x \leq b.$$

3. [**Mattuck Problem 16-4**] This uses the result of Problem 16-3; in it, avoid calculation by using the simple estimation $\sin a \leq a$, $a > 0$.

1. You are given values of $\sin x$ for $x = .05, .10, .15, \ldots$, and use linear interpolation to calculation $\sin .12$ and $\sin .48$. In each case, tell how many decimal places the answer is correct to; explain geometrically why one answer is more accurate than the other.

2. Over approximately what interval $[0, a]$ could you be sure that the linear interpolation process for $\sin x$ would be accurate to within .0001?

4. [**Mattuck 17.3/2,3,5**] (17.3/2) You wish to use $1 - x^2/2$ as an approximation to $\cos x$, with an error not greater than .0001. Estimate over what interval this will be valid.

(17.3/3) You want to estimate $\sin x$ to three decimal places over $|x| < .5$. How large should n be in order that the n-th order Taylor polynomial give you this accuracy over the given interval? Show your answer is correct.

(17.3/5) Calculate $\ln 1.1$ to three decimal places by using a suitable Taylor polynomial, and show your answer has the desired accuracy.

5. [**Mattuck 17.4/1**] Prove the Taylor series at 0 of the following functions converge to the function for all x, or for the indicated values of x:
(a) $\sin x$
(b) $\cos x$
(c) $\frac{1}{1-x}$, $x \in (-1, 0]$ll
(d) $\ln(1 + x)$, $x \in [0, 1)$.

(For the last two, explicit remainders were given in Section 4.2. Here, use the general form of the remainder, as given in (11).)

6. [**Mattuck Problem 17-1**] A polynomial $P(x)$ has a **k-fold zero** at the point a if it has $(x-a)^k$ as a factor, but not $(x-a)^{(k+1)}$, i.e.,

$$P(x) = (x-a)^k Q(x),$$

where $Q(x)$ is a polynomial such that $Q(a) \neq 0$.

1. Prove: the point a is a k-fold zero for the polynomial $P(x)$ if and only if
$$P(a) = P'(a) = \cdots = P^{(k-1)}(a) = 0, \quad P^{(k)}(a) \neq 0.$$

2. For what value(s) of the constant b will $2x^3 - bx^2 + 1$ have a double zero at some point (i.e., a 2-fold zero)?

3. Using part (a), describe the connection between Exercise 17.2/3 and the Extended Rolle's Theorem (Lemma 17.2).