MATH 3110: Homework #12 (due Thursday, December 1)

November 22, 2016

The references to [**Mattuck**] refer to the **Exercises** at the end of chapter. If you'd like to receive full credit, be sure to show your reasoning. Try to write in complete sentences.

1. [**Mattuck 22.1/2, 22.2/1,3**] (22.1/2) Prove that the Taylor series for $\sin x$ converges uniformly in any interval $[-R, R]$, but does not converge uniformly on $(-\infty, \infty)$.

(Don't quote any theorems presented in later sections of the chapter; do it directly from the definition of uniform convergence. Model the argument after that in Example 22.1E, using the remainder term in the Taylor series.)

(22.2/1) Prove that $f_n \overset{unif}{\to} f$ on I if and only if $\sup_I |f(x) - f_n(x)| \to 0$.

(22.2/3) (a) Show the geometric series $\sum x^n$ converges uniformly in every interval $[-L, L]$, for $0 < L < 1$.

(b) Show that it does not converge uniformly in $(-1, 1)$, by considering the explicit remainder term for the series (Section 4.2 (4)).

2. [**Mattuck 22.2/5, 22.3/4**] (22.2/5) Prove the Cauchy criterion for uniform convergence, given below. Unlike the elementary criterion (Theorem 22.2A), it does not require advance knowledge of the limit $f(x)$.

> **Cauchy criterion for uniform convergence.** Suppose that on an interval I, the functions $f_n(x)$ are defined, and given $\epsilon > 0$, we have $|f_m(x) - f_n(x)| < \epsilon$ for $m, n \gg 1$, i.e. for $m, n > N_\epsilon$ for some N_ϵ depending only on ϵ. Then the sequence $f_n(x)$ convergences uniformly on I.

(The Cauchy criterion for convergence of numerical sequences (6.4) will be a big help in the argument, but its proof will not be.)

(22.3/4) (a) Prove that if the functions $f_n(x)$ are uniformly continuous on I, and $f_n(x) \overset{unif}{\to} f(x)$ on I, then $f(x)$ is uniformly continuous on I.

(b) How do the functions

$$f_n(x) = \begin{cases} n, & 0 < x < 1/n; \\ 1/x, 1/n \le x \le 1; \end{cases}$$

for $n \in \mathbf{N}$, illustrate the theorem in part (a)? (Do they satisfy the hypotheses? Does their limit $f(x)$ satisfy the conclusion?)

3. [**Mattuck 22.4/4,5**] (22.4/4) (a) Prove $f(x) = \sum_1^\infty \frac{x^{n-1}}{n^2(1+x^n)}$ is continuous for $x \ge 0$.

(b) Evaluate $\int_0^1 f(x)\,dx$, justifying all the steps of your work, and expressing your answer in terms of values of the function $\zeta(s) = \sum_1^\infty \frac{1}{n^s}$.

(22.4/5) Let $Z_p(u)$ be the function defined by the series $Z_p(u) = \sum_1^\infty \frac{u^n}{n^p}$. Evaluate $\int_0^1 Z_2(e^{-x})\,dx$ in terms of values of $Z_p(u)$, justifying all steps.

4. [**Mattuck 22.6/3,4**] (22.6/3) Let $\sum a_n x^n$ be the power series solution to the differential equation $y' - y = e^x$, satisfying the initial condition $y(0) = 0$.

(a) Use the Uniqueness Theorem (Cor. 22.6C) to find the recursion relation connecting a_{n+1} and a_n.

(b) Calculate the first few values of a_n, guess what a_n will be in general, and prove it by induction (Appendix A.4).

(c) Find the sum of the power series.

(22.6/4) Show that for any power series $\sum a_n x^n$ with a positive radius of convergence, there is a function $f(x)$ whose Taylor series is $\sum a_n x^n$, yet for which $f(x) \ne \sum a_n x^n$ for $x \approx 0$.

5. [**Mattuck Problem 22-1**] Find the sum of $\sum_{n=1}^\infty \frac{x^n}{n^2}$, expressed as the integral of an elementary function. Justify the steps, including the existence of the integral. On what interval is the representation valid, and why?

6. [**Mattuck Problem 22-4**] Prove that if $f(x) = e^{-1/x^2}$, (and defining $f(0) = 0$), then $f(x)$ has derivatives of all orders at $x = 0$, and $f^{(n)}(0) = 0$ for all n.

(Try the first few derivatives, to see the pattern. In addition to the usual differentiation rules, you will have to use the definition of the derivative, as well as l'Hospital's rule.)