MATH 3110: Homework #3
(due Tuesday, September 13)

September 6, 2016

If you'd like to receive full credit, be sure to show your reasoning. Try to write in complete sentences.

1. [**Mattuck 3.2/2**] Suppose $\{a_n\}$ is a convergent increasing sequence, and $\lim a_n = L$. Let $\{b_n\}$ be another sequence "interwoven" with the first, i.e., such that

$$a_n < b_n < a_{n+1} \quad \text{for all } n.$$

Prove from the definition of limit that $\lim b_n = L$ also.

2. [**Mattuck 3.2/3**] (a) Prove the sequence

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

has a limit.

(b) Criticize the following "proof" that its limit is 0: Given $\epsilon > 0$, then for $i = 1, 2, 3, \ldots$, we have

$$\frac{1}{n+i} < \epsilon, \quad \text{if } \frac{1}{n} < \epsilon, \quad \text{i.e., if } n > 1/\epsilon.$$

Adding up these inequalities for $i = 1, \ldots, n$ gives

$$0 < a_n < n\epsilon, \quad \text{for } n > \frac{1}{\epsilon};$$

Therefore, $|a_n - 0| < n\epsilon$ for $n \gg 1$. By the definition of limit and the K-ϵ principle, $\lim a_n = 0$.

3. [**Mattuck Problem 3-1**] Let $\{a_n\}$ be a sequence and $\{b_n\}$ be its sequence of averages:

$$b_n = (a_1 + \cdots + a_n)/n \quad \text{(cf. Problem 2-1)}.$$

(a) Prove that if $a_n \to 0$, then $b_n \to 0$. (Hint: this uses the same ideas as example 3.7. Given $\epsilon > 0$, show how to break up the expression for b_n into two pieces, both of which are small, but for different reasons.)

(b) Deduce from part (a) in a few lines without repeating the reasoning that if $a_n \to L$, then also $b_n \to L$.

4. [**Mattuck Problem 3-4**] Prove that a convergent sequence $\{a_n\}$ is bounded.

5. [**Mattuck 4.3/3**] Using Newton's method to find M, the unique positive zero of $x^2 + x - 1$:

(i) Give the recursion formula for a_{n+1} in terms of a_n.

(ii) Obtain a recursion formula for the error term $e_n = a_n - M$. Use it to prove $a_n \to M$, if a_0 lies in a suitable interval. (Hint: in the algebraic calculations, it is best to leave M as a letter; at a certain point you can simplify the expressions by using $M^2 + M - 1 = 0$.)

(iii) Section 4.4 describes another sequence that converges to M. Read the conclusion of section 4.4; which sequence converges faster to M, and why?

6. [**Mattuck Problem 4-2**] Pick a positive number between 0 and $\pi/2$, take its cosine, then take the cosine of that number, and keep on taking cosines. You get a sequence $\{a_n\}$ given by $a_{n+1} = \cos a_n$.

(a) Try it on a calculator a few times. (**Make sure it's in radians!**) What eight place decimal number L do you end up with? What equation is it a root of?

(b) Prove the sequence converges to this limit, by studying the error term and showing it tends to 0 in the limit; cf. the examples in this chapter. (Use the estimations $|1 - \cos x| \leq x^2/2$ and $|\sin x| \leq |x|$, valid for all x.)