MATH 3110: Homework #4
(due Tuesday, September 20)

September 13, 2016

The references to [**Mattuck**] refer to the **Exercises** at the end of chapter. If you'd like to receive full credit, be sure to show your reasoning. Try to write in complete sentences.

1. [**Mattuck 5.4/1**] Suppose the terms of the sequence $\{a_n\}$ are colored, using k different colors (red, blue, yellow, etc.), and using each color infinitely often. Then we get subsequences: $\{a_{r_i}\}$ = all the red terms; $\{a_{b_i}\}$ = all the blue terms; etc.

Prove: if each of these k subsequences converges to the same limit L, then $\{a_n\}$ converges to L. (This is a partial converse to the Subsequence Theorem.)

1*. (Optional bonus problem, 3 pts) [**Mattuck Problem 5-3**] Show by counterexample that the partial converse to the Subsequence Theorem given in Exercise 5.4/1 would be false if we allowed k to be ∞. That is, if we divide up $\{a_n\}$ into an infinity of infinite subsequences, each of which converges to L, it does not follow that the sequence $\{a_n\}$ itself converges to L.

2. [**Mattuck 5.4/2**] Let $s(n)$ be the sum of the prime factors of the integer n. For example, $6 = 2 \cdot 3$, so $s(6) = 5$; similarly, $8 = 2^3$, so $s(8) = 6$.

Prove that $\lim_{n\to\infty} \left(\frac{s(n)}{n}\right)$ does not exist. (Use Thm 5.4 and Exer. 3.4/4.)

3. [**Mattuck Problem 5-7**] Define a sequence recursively by $a_{n+1} = \sqrt{2a_n}$, $a_0 > 0$.

(a) Prove that for any choice of $a_0 > 0$, the sequence a_n is monotone and bounded.

(b) Part (a) shows the limit L exists; determine L, prove it is the limit, and is independent of the choice of $a_0 > 0$.

4. [**Mattuck 6.4/1,2,3**] (1) Prove that every convergent sequence is a Cauchy sequence.

(2) Suppose a sequence $\{a_n\}$ has this property: there exist constants C and K, with $0 < K < 1$, such that

$$|a_n - a_{n+!}| < CK^n,$$

for $n \gg 1$. Prove that $\{a_n\}$ is a Cauchy sequence.

(3) Show that $\sqrt{n}$ is another example which illustrates Question 6.4/1: a sequence whose successive terms get arbitrarily close is not necessarily a Cauchy sequence.

5. [**Mattuck Problem 6-6**] (lim sup and lim inf) For bounded sequences which do not converge, these are notions which sometimes can substitute for the non-existent limit.

Let $\{a_n\}$ be a bounded sequence. Let $T_n = \{a_i : i \geq n\}$ denote the n-th "tail" of the sequence, and define

$$\overline{b_n} = \sup T_n \quad \text{and} \quad \underline{b}_n = \inf T_n.$$

(a) Prove the sequence $\{\overline{b}_n\}$ and $\{\underline{b}_n\}$ both converge.

We now define

$$\limsup a_n = \lim \overline{b_n}, \quad \liminf a_n = \lim \underline{b}_n.$$

(b) Find lim sup and lim inf for the sequence $a_n = \frac{1}{n} + (-1)^n$.

(c) Prove that $\liminf a_n \leq \limsup a_n$.

(d) Prove that $\lim a_n$ exists $\Leftrightarrow \liminf a_n = \limsup a_n$.

6. [**Mattuck Problem 6-7**] Continuing Problem 6-6, let S denote the set of cluster points of $\{a_n\}$. Prove

$$\limsup a_n = \max S, \quad \liminf a_n = \min S.$$

(Note: it is not obvious that $\min S$ and $\max S$ even exist. The problem requires you to show, among other things, that $\limsup a_n$ and $\liminf a_n$ are actually themselves cluster points of the sequence.

The result of this problem is probably the most intuitive way to think about lim sup and lim inf—they are respectively the highest and lowest cluster points of the sequence. Such cluster points exist by the Cluster Point Theorem and the Bolzano–Weierstrass Theorem, since $\{a_n\}$ is bounded.)