MATH 3110: Homework #5
(due Tuesday, September 27)

September 20, 2016

The references to [**Mattuck**] refer to the **Exercises** at the end of chapter. If you'd like to receive full credit, be sure to show your reasoning. Try to write in complete sentences.

1. [**Mattuck Problem 6-2**] (a) Let S be a bounded non-empty set of $\mathbf{R}$, and $\overline{m} = \sup S$. Prove there is a sequence $\{a_n\}$ such that $a_n \in S$ for all n, and $a_n \to \overline{m}$.

(You must show how to construct the sequence a_n. Use the properties sup-1 and sup-2 which characterize $\overline{m}$.)

(b) Let A and B be bounded non-empty subsets of $\mathbf{R}$. Prove the equality $\sup(A+B) = \sup A + \sup B$. (cf. Exercise 6.5/3g **(Prove this!)**; use part (a))

2. [**Mattuck Problem 6-3**] Suppose $f(x)$ is continuous and decreasing on $[0, \infty]$, and $f(n) \to 0$. Define $\{a_n\}$ by

$$a_n = f(0) + f(1) + \cdots + f(n-1) - \int_0^n f(x)\, dx.$$

(a) Prove $\{a_n\}$ is a Cauchy sequence directly from the definition.

(b) Evaluate $\lim a_n$ if $f(x) = e^{-x}$.

3. [**Mattuck Problem 6-5**] Here is a proof of the Bolzano–Weierstrass Theorem which produces a convergent subsequence directly, without first having to find a cluster point. But the first part is a bit subtle.

(a) Prove that every sequence $\{a_n\}$ has a monotonic subsequence.

Method: let $T_n = \{a_i : i \geq n\}$ denote the n-th "tail" of the sequence.

Case 1. If some tail has no maximum, deduce that $\{a_n\}$ has an increasing subsequence.

Case 2. If every tail has a maximum, deduce that $\{a_n\}$ has a decreasing subsequence, by selecting carefully from $\{\max T_n : n \in \mathbf{N}\}$.

(b) Deduce the Bolzano–Weierstrass theorem from part (a).

4. [**Mattuck 7.3/3,4,5**]

3(a). Let $\{a_n\}$ be a sequence, and $\{a_{n_i}\}$ be any subsequence. Prove that if $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, then $\sum_{i=0}^{\infty} a_{n_i}$ is absolutely convergent.

3(b). Show by counterexample that the above is false if the word "absolutely" is dropped everywhere.

4(a). Prove: $\sum a_n$ absolutely convergent $\Rightarrow \sum a_n^2$ convergent.

5(a). How would you define "absolute divergence" of a series $\sum a_n$?

5(b). "If $\sum a_n$ diverges absolutely, then it diverges."

5. [**Mattuck Problem 7-1**] Here is a refinement of the n-th term test for divergence (7.2A).

> **Theorem.** *If $\{a_n\}$ is non-negative, decreasing, and $\sum a_n$ converges, then*
>
> $$na_n \to 0.$$

(a) Illustrate the use of this theorem on the series $\sum 1/n^p$, for $p > 0$. What does it tell you about the convergence or divergence of this series for different values of p? Equally important, what does it *not* tell you?

(b) A converse is: $na_n \to 0$, $a_n \geq 0$, a_n *decreasing* $\Rightarrow \sum a_n$ *converges.* The converse is *not* true. Show this by giving a counterexample.

(c) Prove the theorem. Show first that

$$\text{given } \epsilon > 0, \quad na_{2n} \leq a_{n+1} + \cdots + a_{2n} < \epsilon, \qquad \text{for } n \gg 1.$$

Then handle the odd terms somehow, and complete the proof. (Hint: Exercises 7.1/2 and 6.4/1 may be helpful in this problem.)

6. [**Mattuck Problem 8-1**] Determine, with proof, the radius of convergence of $\sum(\sin n)x^n$. (This is an example of a power series to which the ratio test does not apply; cf. Exercise 8.1/3. Note that there are two things you have to show: one is harder than the other. The end of Chapter 5 may be of help.)