MATH 3110: Homework #6
(due Thursday, October 13)

October 4, 2016

The references to [**Mattuck**] refer to the **Exercises** at the end of chapter. If you'd like to receive full credit, be sure to show your reasoning. Try to write in complete sentences.

1. [**Mattuck 9.3/1**] Suppose $f(x)$ is defined for all x (it would be enough to assume it has a domain D symmetric about 0).

(a) Prove that $E(x) = \frac{f(x)+f(-x)}{2}$ is an even function.

(b) Show $f(x)$ can be expressed as the sum of $E(x)$ and an odd function $O(x)$.

(c) Show that the representation in (b) is unique; that is, if the function f has a representation $f(x) = E_1(x) + O_1(x)$, then $E = E_1$ and $O = O_1$.

(d) How does this representation look if $f(x)$ is a polynomial?

(e) How does it look if $f(x) = e^x$? (If you have met these functions before, now you know where they come from.)

2. [**Mattuck 10.1/2(a), 10.1/8, 10.2/1**]

(10.1/2(a)) Examine the boundedness, sup, inf, max, and min (on D_f) of $1/(1+x^2)$.

(10.1/8) Estimate the value of $\int_0^1 \frac{x^4}{1+x^6}\,dx$ by estimating the integrand, and prove the value of the integral is $\leq 1/5$.

(10.2/1) If you approximate $\ln(1+x)$ by using the first three terms of the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

estimate the error when $0 \leq x \leq .1$. (Use Cauchy's test; cf. Section 7.6 (15).)

3. [**Mattuck Problem 10-1**] [This problem is sort of like a calculus review on steroids, in that probably the simplest way to proceed is to dust off your calculus skills to solve it.] Let

$$f(x) = \int_0^x \frac{\cos t}{1+t^2}\, dt;$$

it is not an elementary function.

(a) Is it odd, even, or neither? (Indicate reason.)

(b) Is it increasing, decreasing, or neither? (Indicate reason.)

(c) Using its geometric interpretation as area, show intuitively that it has a maximum, tell where it occurs, and estimate its maximum value roughly.

4. [**Mattuck Problem 10-3**] Prove that a function which is locally increasing on an interval I is increasing on I. (Suggestion: use an indirect argument and bisection.)

5. [**Mattuck 11.4/2, 11.5/1**] 11.4/2: If $f(x)$ is continuous on $[a, b]$ and strictly increasing on (a, b), prove it is strictly increasing on $[a, b]$.

11.5/1: (a) Prove that if $f(x)$ is continuous, and $f(x) = 0$ when x is a rational number, then $f(x) = 0$ for all x.

(b) Prove that if $f(x)$ and $g(x)$ are continuous, and $f(x) \leq g(x)$ when x is a rational number, then $f(x) \leq g(x)$ for all x. Show that $\leq$ cannot be replaced by $<$ throughout.

6. [**Mattuck Problem 11-1**] Suppose $f(x)$ is continuous for all x and $f(a + b) = f(a) + f(b)$ for all a and b. Prove that $f(x) = Cx$, where $C = f(1)$, as follows:

(a) prove, in order, that it is true when $x = n$, $1/n$, and m/n, where m, n are integers, $n \neq 0$.

(b) use the continuity of f to show it is true for all x.