MATH 3110: Homework #9 (due Tuesday, November 1)

October 25, 2016

The references to [**Mattuck**] refer to the **Exercises** at the end of chapter. If you'd like to receive full credit, be sure to show your reasoning. Try to write in complete sentences.

1. [**Mattuck Problem 13-6**] (a) Prove that if $f(x)$ is uniformly continuous on I, and a_n is a Cauchy sequence in I, then $f(a_n)$ is a Cauchy sequence.

(b) Use part (a) to prove that $1/x$ is not uniformly continuous on $(0, 1]$.

2. [**Mattuck 14.1/5, 6**] (14.1/5) Let $f(x) = x\sin(1/x)$, $x \neq 0$; $f(0) = 0$. Prove that $f(x)$ is continuous at 0, but not differentiable at 0.

(14.1/6) (a) Prove: if $f'(a)$ exists, then $f(x) = L(x) + e(x)$, where

$$L(x) = f(a) + f'(a)(x - a) \quad \text{and} \lim_{x\to a} \frac{e(x)}{x-a} = 0.$$

(b) $L(x)$ is called the *linearization of* $f(x)$ *at* a; explain part (a) geometrically by interpreting $f(x)$, $L(x)$, and $e(x)$ graphically.

3. [**Mattuck Problem 14-1**] Suppose $f(x)$ is differentiable at a. Define a new function by

$$F(\Delta x) = \frac{f(a + \Delta x) - f(a - \Delta x)}{2\Delta x}, \quad x = a + \Delta x.$$

(a) Find $\lim_{\Delta x\to 0} F(\Delta x)$, and interpret F geometrically.

(b) Can the limit of part (a) exists even if f is not differentiable at a? If so, give an example; if not, prove it.

(c) Suppose the right- and left-hand derivatives of $f(x)$ exist at a, but they are not equal. What is $\lim_{\Delta x \to 0} F(\Delta x)$? Prove it.

4. [**Mattuck Problem 14-7**] Prove the Chain Rule 14.3B for the general case (i.e., without the assumption $\Delta x \neq 0$ for $\Delta t \underset{\neq}{\approx} 0$), as follows.

(a) Show the hypothesis of differentiability can be written in the form of the two equations (cf. Exercise 14.1/6)

$$\Delta y = f'(x_0)\Delta x + e_1, \quad \Delta x = g'(t_0)\Delta t + e_2;$$

where

$$\lim_{\Delta x \to 0} \frac{e_1}{\Delta x} = 0, \quad \lim_{\Delta t \to 0} \frac{e_2}{\Delta t} = 0.$$

(b) Substitute the second equation into the first, to get

$$\Delta y = f'(x_0)g'(t_0)\Delta t + e,$$

where the error term e is expressed in terms of the error terms e_1 and e_2.

(c) Then use the expression for e to show that

$$\lim_{\Delta t \to 0} \frac{e}{\Delta t} = \lim_{\Delta t \to 0} \epsilon \cdot \frac{\Delta x}{\Delta t},$$

where

$$\epsilon = \begin{cases} e_1/\Delta x, & \Delta x \neq 0, \\ 0, & \Delta x = 0; \end{cases}$$

then show this limit is 0 and finish the argument using (b).

(The simple proof of the chain rule we gave before used Δx in the denominator of a fraction, so we had to assume it was non-zero. By using here the linearizations and introducing the new function ϵ, we avoid ever having to write Δx in a denominator.)

5. [**Mattuck Problem 15-4**] Prove l'Hospital's rule for the case ∞/∞ (as $x \to \infty$) along the following lines.

Let $L = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$;, choose a so that $\left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon$ for $x > a$.

Prove the two approximations below (valid for $x \gg 1$); deduce Theorem 14.5C:

$$\left|\frac{f(x)}{g(x)} - \frac{f(x) - f(a)}{g(x) - g(a)}\right| < \epsilon$$

$$\left|\frac{f(x) - f(a)}{g(x) - g(a)} - L\right| < \epsilon.$$

(Hint: for the first, write $f(x) - f(a) = f(x)[1 - f(a)/f(x)]$, and use limit theorems; for the second, use the Cauchy Mean-value Theorem.)

6. [**Mattuck Problem 15-5**] Suppose $f(x)$ is differentiable on an interval I (which may be open), and $f'(x)$ is bounded on I. Prove that $f(x)$ is uniformly continuous on I.

(Give a direct proof, based on the ideas of this chapter and definition of uniform continuity. Note that the Uniform Continuity Theorem 13.5 is not applicable, since the interval I is not assumed compact.)