HW #11 solutions

1. (HW 6.50)

i) $e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y))$

$u = e^x\cos(y)$; $v = e^x\sin(y)$ (not conformal)

u constant: $e^x = u\sec(y)$

v constant: $e^x = v\csc(y)$

ii) $\frac{z-1}{z} = 1 - \frac{1}{z} = 1 - \frac{1}{x+iy} = 1 - \frac{x-iy}{x^2+y^2}$

$u = \frac{x^2+y^2-x}{x^2+y^2}$

$V = \frac{y}{x^2+y^2}$

constant u: $(x^2+y^2)(u-1) + x = 0$

$x^2+y^2 + \frac{x}{u-1} = 0$ $\quad u \neq 1$

$\left(x + \frac{1}{2(u-1)}\right)^2 + y^2 = \frac{1}{4(u-1)^2}$, circle of radius $\frac{1}{2(u-1)}$ centered at $\frac{-1}{2(u-1)}$

constant v: $x^2+y^2 - \frac{y}{v} = 0$

$x^2 + \left(y - \frac{1}{2v}\right)^2 = \frac{1}{4v^2}$ circle of radius $\left|\frac{1}{2v}\right|$ centered at $\frac{i}{2v}$

iii) on main branch, $z = re^{i\theta}$ $\quad 0 < \theta < 2\pi$

$-1 + \ln(z) = i\theta + \ln(r) - 1$ $\quad u = \ln(r) - 1 = \ln(\sqrt{x^2+y^2}) - 1$

$v = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

u constant $\to$ $\sqrt{x^2+y^2}$ constant $\to$ circles centered at $z=0$, of radius e^{u+1}.

v " $\to$ $\frac{y}{x}$ constant $\to$ line $y = ax$.

2 (KW 6.51)

i) $w(z) = \frac{z-1}{z+1}$ $\quad |z-1| \leq |z+1|$

z is closer to 1 than it is to -1. → right half-plane

ii) $|z+i| \geq |z-i|$ $\quad$ z is closer to i than it is to $-i$ → upper half-plane.

3. (KW 6.54).

$$w(z) = \frac{z^2+1}{z} = \frac{x^2-y^2+1+2ixy}{x+iy}$$

$$= \frac{1}{x^2+y^2}\left((x^2-y^2+1+2i(xy))(x-iy)\right)$$

$$= \frac{1}{x^2+y^2}\left(x^3-y^2x+x+2ix^2y-ix^2y+iy^3-iy+2xy^2\right)$$

$$= \frac{1}{x^2+y^2}\left(x^3+y^2x+x+i\cdot(x^2y+y^3-y)\right)$$

$$v = \frac{x^2y+y^3-y}{x^2+y^2}$$

$$u = \frac{x^3+y^2x+x}{x^2+y^2}$$

a), b) → see plots.

c) $\bar{v} = -p\,\bar{\nabla}P(x,y)$

if u corresponds to pressure,

$$P(x,y) = u(x,y)$$

$$\bar{\nabla}P(x,y) = \nabla u = \left(\frac{x^4+y^4+y^2+x^2(2y^2-1)}{(x^2+y^2)^2},\ \frac{-2xy}{(x^2+y^2)^2}\right)$$

$$= \left(1+\frac{y^2-x^2}{x^2+y^2},\ \frac{-2xy}{x^2+y^2}\right)$$

see plot in pptx. file.

4. (KW 6.55).

at -1, $\arg\left(\frac{dz}{dw}\right)$ decreases by $\frac{\pi}{2}$

at 1, $\arg\left(\frac{dz}{dw}\right)$ increases by $\pi/2$

$$\frac{dz}{dw} = A\,(w-(-1))^{\frac{1}{2}}\,(w-1)^{-\frac{1}{2}}$$ A will be real.

$$\boxed{\frac{dz}{dw} = A\,(w+1)^{\frac{1}{2}}\,(w-1)^{-\frac{1}{2}}}$$

$$\frac{dz}{dw} = A\sqrt{\frac{w+1}{w-1}}$$ if $w > -1$

$$dz = A\sqrt{\frac{w+1}{w-1}}\,dw$$

$$z(w) = A\int\sqrt{\frac{w+1}{w-1}}\,dw = A\int\frac{w}{\sqrt{w^2-1}} + \frac{1}{\sqrt{w^2-1}}\,dw$$

$$= A(w^2-1)^{1/2} + A\ln\left(\sqrt{w^2-1}+w\right) + C$$

$$z(1) = C = 0 \rightarrow \boxed{C=0}$$

$$z(-1) = A\ln(-1) = ih \rightarrow A\cdot(2n+1)i\pi = ih \rightarrow \boxed{A = \frac{h}{(2n+1)\pi}} \quad n\in\mathbb{Z}$$

constant $v \rightarrow w = u+iv$

say $n=0$. $A = \frac{h}{\pi}$

$$z(w) = \frac{h}{\pi}\left((u+iv)^2-1\right)^{1/2} + \frac{h}{\pi}\ln\left(((u+iv)^2-1)^{1/2} + (u+iv)\right)$$

5. (KW 6.57)

$$\frac{dz}{dw} = A(w-w_1)^{-k_1}(w-w_2)^{-k_2}(w-w_3)^{-k_3}\ldots$$

find inverse, plug in

a) $e^{i\pi/4}(w+1)^{-\frac{1}{2}}(w)^{1}(w-1)^{-\frac{1}{2}} = \frac{dz}{dw}$

$\arg\left(\frac{dz}{dw}\right) = \arg(A) - (k_1+k_2+k_3)\pi$ for $w < -1$

$= \arg(A) = \boxed{\pi/4}$

$-1<w<0$: $\arg\left(\frac{dz}{dw}\right) = \arg(A) - (k_2+k_3)\pi = \arg(A) - \left(\frac{-1}{2}\right)\pi = \frac{\pi}{4}+\frac{\pi}{2} = \boxed{\frac{3\pi}{4}}$

$0<w<1$: $\arg\left(\frac{dz}{dw}\right) = \arg(A) - k_3\pi = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

$1<w$: $\arg\left(\frac{dz}{dw}\right) = \arg(A) = \frac{\pi}{4}$

to find actual points:

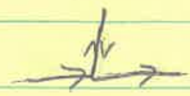
$$\frac{dz}{dw} = e^{i\pi/4}(w+1)^{-\frac{1}{2}}(w)^{1}(w-1)^{-\frac{1}{2}} = \frac{w}{\sqrt{w^2-1}}$$

$z(w) = e^{i\pi/4}\sqrt{(w+1)(w-1)} + C$ C is arbitrary; say $C=0$

$z(w) = e^{i\pi/4}\sqrt{(w+1)(w-1)}$ $z(\pm 1) = 0$

$z(0) = e^{i3\pi/4}$

b) this is just rotated by $\pi/4$ to the right:

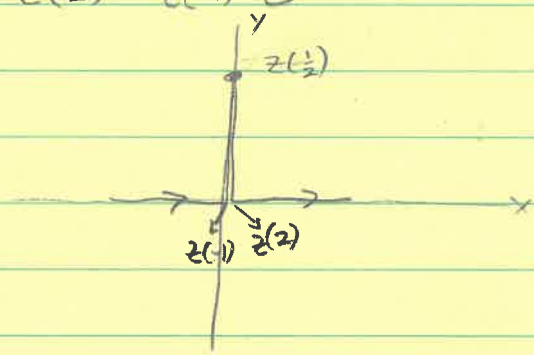


$$\frac{dz}{dw} = (w+1)^{-1/2}(w-\tfrac{1}{2})^{1}(w-2)^{-1/2}$$

$$\frac{dz}{dw} = \frac{w-\frac{1}{2}}{\sqrt{(w+1)(w-2)}} \longrightarrow z = \sqrt{(w-2)(w+1)} + C$$

again, C is arbitrary; we'll set ~~it~~ to 0.

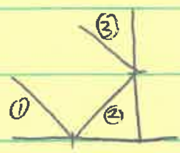
$z(2) = z(-1) = 0$



6. (KW 6.58)

$\frac{dz}{dw} = i w^{-\frac{1}{2}} a$ $\quad$ a = any nonzero real number (positive written above "nonzero")

$z = a 2 i \sqrt{w} \rightarrow \sqrt{w} = \frac{z}{2ia} \rightarrow \boxed{w = \frac{-z^2}{4a^2}}$



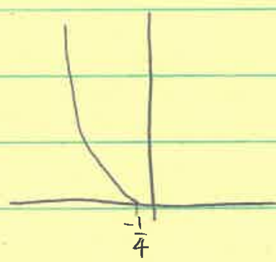
segment ①: $y = -x - 1$ $\quad$ use $a = 1$

$z = t + i(-t-1)$ $\quad$ $\underline{\underline{-\infty < t < -1}}$

$w = \frac{-1}{4} z^2 = \frac{-1}{4}\left[(-1-2t) + i(-2t - 2t^2)\right]$

$w = \frac{1}{4}(1+2t) + i \frac{1}{2}(t + t^2)$

① $\rightarrow$

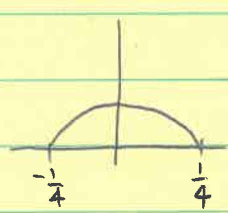


②: $y = x+1 \rightarrow x = t,\ y = t+1$ $\quad$ $-1 < t < 0$

$z = t + i(t+1)$

$w = \frac{-1}{4} z^2 = \frac{-1}{4}\left(-1-2t + i(2t + 2t^2)\right)$

$w = \frac{1}{4}(1+2t) + i\left(-\frac{t}{2} - \frac{t^2}{2}\right)$



6. contd ③: $y = -x + 1$ $\qquad 0 < t < -\infty$

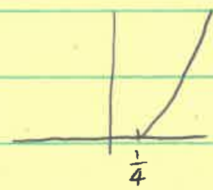
$x = t$

$y = -t + 1$

$z = t + i(1-t)$

$w = -\frac{1}{4}z^2 = -\frac{1}{4}\left(2t - 1 + i(2t - 2t^2)\right)$

$w = \frac{1}{4}(1-2t) + i\frac{1}{2}(t^2 - t)$



so we get