AEP 4210
HW #12 Solutions

1. (KW 7.4) $T = 4$, say amplitude $= A$

i) take the interval between $t=0$, $t=4$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T}t\right)$$

$$a_n = \frac{2}{T}\int_0^T dt\, f(t)\cos\left(\frac{2\pi n}{T}t\right) = \frac{2}{4}\int_0^4 f(t)\cos\left(\frac{2\pi n}{T}t\right)dt = \frac{2A}{4}\int_0^1 \cos\left(\frac{\pi n t}{2}\right)dt$$

$$= \frac{A}{2}\left(\frac{2}{\pi n}\right)\sin\left(\frac{\pi n t}{2}\right)\Big|_0^1 = \boxed{\frac{A}{\pi n}\sin\left(\frac{\pi}{2}n\right) = a_n}$$

$$b_m = \frac{2}{T}\int_0^T dt\, f(t)\sin\left(\frac{2\pi m}{T}t\right) = \frac{A}{2}\int_0^1 \sin\left(\frac{\pi m t}{2}\right)dt = \boxed{\frac{-A}{\pi n}\left(\cos\left(\frac{\pi n}{2}\right)-1\right) = b_n}$$

$$\frac{a_0}{2} = \frac{1}{T}\int_0^1 A\,dt = \frac{A}{T} = \frac{A}{4}$$

$\sin\left(\frac{\pi}{2}n\right) = 0$ if $n = 2k$ $k \in \mathbb{Z}$

1 if $\frac{\pi}{2}n = 2\pi k + \frac{\pi}{2} \rightarrow n = 4k+1$

-1 if $\frac{\pi}{2}n = 2\pi k - \frac{\pi}{2} \rightarrow n = 4k-1$

$\cos\left(\frac{\pi}{2}n\right) = 0$ if $n = 2k+1$; 1 if $n = 4\pi k$; -1 if $n = 4k+2$

ii) $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}$ $\omega_n = \frac{2\pi n}{T} = \frac{\pi n}{2}$

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-i\omega_n t}dt = \frac{1}{4}\int_0^4 f(t)\, e^{-i\frac{\pi n t}{2}}dt = \frac{A}{4}\int_0 e^{-i\frac{\pi}{2}nt}$$

$$= \frac{A}{4}\left(\frac{2i}{\pi n}\right)e^{-\frac{i\pi n t}{2}}\Big|_0^1 = \frac{Ai}{2\pi n}\left(e^{-\frac{i\pi n}{2}} - 1\right)$$

$$\boxed{c_n = \frac{Ai}{2\pi n}\left(\cos\left(\frac{\pi n}{2}\right)-1\right) + \frac{A}{2\pi n}\sin\left(\frac{\pi n}{2}\right)}$$

check against formuli 7.32 on page 2.28

$n=0$: $c_0 = \frac{A}{4} = \frac{a_0}{2}$ ✓

$n>0$: $c_n = \frac{Ai}{2\pi n}\left(\cos\left(\frac{\pi n}{2}\right)-1\right) + \frac{A}{2\pi n}\sin\left(\frac{\pi n}{2}\right) = \frac{1}{2}A_n - \frac{1}{2}i\,b_n$ ✓

$n<0$: $\frac{a_{-n}+ib_{-n}}{2} = \frac{1}{2}\left[\left(\frac{-A}{\pi n}\sin\left(\frac{\pi}{2}n\right)\right) + i\left(\frac{A}{\pi n}\left(\cos\left(\frac{\pi n}{2}\right)-1\right)\right)\right] = c_n$ ✓

2. (KW 7.5 i))

$T = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} t^3 \cos(2nt)\,dt = \frac{3\pi}{2n^2}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} t^3 \sin(2nt)\,dt = \frac{3}{2n^3} - \frac{\pi^2}{n}$$

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} t^3\,dt = \frac{\pi^3}{4}$$

3. (KW 7.10)

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}$$

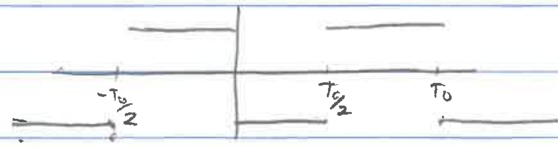
$$\frac{df}{dt} = \sum_{n=-\infty}^{\infty} c_n (i\omega_n) e^{i\omega_n t}$$

so $C_n = i\omega_n c_n$

$\Omega_n = \omega_n$

4. (KW 7.12)

a) $f(t) = \begin{cases} -1 & 0 < t < T_0/2 \\ 1 & T_0/2 < t < T_0/2 \end{cases}$



$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi}{T_0}nt}$$

$$c_n = \frac{1}{T_0}\int_0^{T_0} f(t)e^{-i\frac{2\pi n t}{T_0}}dt = \frac{1}{T_0}\left[-\int_0^{T_0/2} e^{-i\frac{2\pi n}{T_0}t}dt + \int_{T_0/2}^{T_0} e^{-i\frac{2\pi n}{T_0}t}dt\right]$$

$$= \frac{1}{T_0}\left[\frac{T_0}{i2\pi n}\left(e^{-i\pi n}-1\right) - \frac{T_0}{i2\pi n}\left(e^{-i2\pi n} - e^{-i\pi n}\right)\right]$$

$$= -\frac{i}{2\pi n}\left[e^{-i\pi n} - 1 + (-1)\left(1 - e^{-i\pi n}\right)\right]$$

$$= \frac{-i}{2\pi n}\cdot 2\left(e^{i\pi n} - 1\right) = \frac{i}{\pi n}\left(1 - e^{i\pi n}\right) = \begin{cases} 0 & n = 2k \quad k \in \mathbb{Z} \\ \frac{2i}{\pi n} & n = 2k+1 \end{cases}$$

$$\frac{dx}{dt} + x(t) = \sum_{k=-\infty}^{\infty} \frac{2i}{\pi(2k+1)} e^{i\frac{2\pi}{T_0}(2k+1)t}$$

assume $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{2\pi}{T_0}kt} \rightarrow \dot{x}(t) = \sum_{k=-\infty}^{\infty} i\frac{2\pi}{T_0}k c_k e^{i\frac{2\pi k}{T_0}t}$

$$x(t) + \dot{x}(t) = \sum_{k=-\infty}^{\infty} c_n\left(1 + \frac{2\pi i n}{T_0}\right)e^{i\frac{2\pi n t}{T_0}} = \sum_{-\infty}^{\infty} \frac{2i}{\pi(2k+1)} e^{i\frac{2\pi}{T_0}(2k+1)t}$$

so $c_n = 0$ if n even

$$\boxed{c_n = \frac{2i}{\pi n\left(1 + \frac{2\pi i n}{T_0}\right)}} \quad n = 2k+1$$

d) driven RC circuit. f is square wave actuator; response is a triangle wave