HW#3
solutions

1. (KW 2.8, c)

$\oint_S d\bar{\sigma} \cdot \bar{V}$

S = sphere of radius R, centered at origin

$\bar{V} = (x^5, y^5, z^5)$

$$\oint_S d\bar{\sigma} \cdot \bar{V} = \int_V d\tau \, \bar{\nabla} \cdot \bar{V} = 5\int_V x^4 + y^4 + z^4$$

use spherical coordinates

$$= 5\int_0^R \int_0^{2\pi} \int_0^{\pi} \left((r\sin(\theta)\cos(\varphi))^4 + (r\sin(\theta)\sin(\varphi))^4 + (r\cos(\theta))^4 \right) r^2 \sin(\theta) \, d\theta \, d\varphi \, dr$$

$$= \frac{12R^7\pi}{7}$$

=

2. (KW 2.9)

$\oint d\bar{r} \cdot \bar{V}$

$\bar{V} = (2y^2 + 3x^2y)\,\hat{e}_x + (4xy - x^3)\,\hat{e}_y$

$\oint d\bar{r} \cdot \bar{V} = \int d\bar{\sigma} \cdot (\bar{\nabla} \times \bar{V})$

$\bar{\nabla} \times \bar{V} = \left(\frac{\partial}{\partial x} V_y - \frac{\partial}{\partial y} V_x\right) \hat{e}_z$

$+ \left(\frac{\partial}{\partial z} V_x - \frac{\partial}{\partial x} V_z\right) e_y$

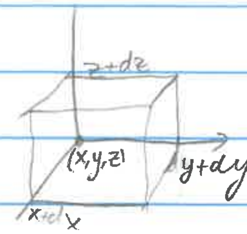
$+ \left(\frac{\partial}{\partial y} V_z - \frac{\partial}{\partial z} V_y\right)$

$= \left(4y - 3x^2 - (4y - 3x^2)\right) \hat{e}_z = \emptyset$

so $\oint_C d\bar{r} \cdot \bar{V} = \int_S d\bar{\sigma} \cdot (0) = \boxed{\emptyset}$

3. (KW 2.12)

a) consider a small cube S.



$d\tau = dx\,dy\,dz$

$$\oint \Phi\, d\vec{\sigma} = \Phi(x+dx,y,z)(dy\,dz,0,0) - \Phi(x,y,z)(dy\,dz,0,0)$$
$$+ \Phi(x,y+dy,z)(dx\,dz,0,0) - \Phi(x,y,z)(dx\,dz,0,0)$$
$$+ \Phi(x,y,z+dz)(dy\,dz,0,0) - \Phi(x,y,z)(dx\,dy,0,0)$$

Taylor expand:

$$\oint \Phi\, d\vec{\sigma} \approx \left(\Phi(x,y,z) + \frac{\partial\Phi}{\partial x}(x,y,z)\,dx - \Phi(x,y,z)\right)(dy\,dz,0,0)$$
$$+ \left(\Phi(x,y,z) + \frac{\partial\Phi}{\partial y}(x,y,z)\,dy - \Phi(x,y,z)\right)(0,dx\,dz,0)$$
$$+ \left(\Phi(x,y,z) + \frac{\partial\Phi}{\partial z}(x,y,z)\,dz - \Phi(x,y,z)\right)(0,0,dx\,dy)$$

$$= \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right)\Big|_{x,y,z} dx\,dy\,dz$$

so $\lim_{S\to 0} \oint \Phi\, d\vec{\sigma} = \int_V d\tau\, \nabla\Phi(x,y,z)$

$\nabla\Phi(x,y,z) \approx$ constant as $V \to 0$.

so $\int_V d\tau\, \nabla\Phi(x,y,z) \approx \nabla\Phi(x,y,z) \int_V d\tau$

$$\therefore \lim_{\substack{S\to 0\\ V\to 0}} \frac{\oint \Phi\, d\vec{\sigma}}{\int_V d\tau} = \nabla\Phi \frac{\int_V d\tau}{\int_V d\tau} = \nabla\Phi, \text{ QED.}$$

3 (KW 2.12)

b) consider the same cube as in part a)

$$\bar{A}\cdot d\bar{\sigma} = (\bar{A}(x+dx,y,z) - \bar{A}(x,y,z))\cdot(dy\,dz,0,0)$$
$$+ (\bar{A}(x,y+dy,dz) - \bar{A}(x,y,z))\cdot(0,dx\,dz,0)$$
$$+ (\bar{A}(x,y,z+dz) - \bar{A}(x,y,z))\cdot(0,0,dx\,dy)$$

$\frac{\partial \vec{\partial}}{\partial x_i}$ exists in $\hat{e}_i$-direction

by the same Taylor-expansion as in a),

$$\bar{A}\cdot d\bar{\sigma} \approx \left(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}\bigg|_{x,y,z}\right) dx\,dy\,dz = \bar{\nabla}\cdot\bar{A}(x,y,z)\,dx\,dy,\,dz$$

so $\oint_S \bar{A}\cdot d\bar{\sigma} = \int_V d\tau\ \nabla\cdot A(x,y,z)$

as $V, S \to \emptyset$, $\int_V d\tau\ \nabla\cdot A(x,y,z) \to \bar{\nabla}\cdot\bar{A}(x,y,z)\int_V d\tau$

so $\dfrac{\oint_S \bar{A}\cdot d\bar{\sigma}}{\int_V d\tau} \to \nabla\cdot A$ QED.

3. (KW 2.12)

c)

$d\bar{\sigma} \times \bar{A}$: consider same cube as in previous 2 examples.

front face: $d\bar{\sigma} = (dydz, 0, 0)$

$$d\bar{\sigma} \times A = (dy\,dz, 0, 0) \times \bar{A}(x+dx, y, z)$$

$$\approx (dydz, 0, 0) \times \left(\bar{A}(x, y, z) + \frac{\partial \bar{A}}{\partial x}(x,y,z)dx\right)$$

back face: $(-dy\,dz, 0, 0) \times \bar{A}(x,y,z)$

front + back faces $= (dy\,dz, 0, 0) \times \frac{\partial \bar{A}}{\partial x} dx$

likewise, right + left faces $= (0, dxdz, 0) \times \frac{\partial \bar{A}}{\partial y} dy$

top + bottom faces $= (0, 0, dxdy) \times \frac{\partial A}{\partial z} dz$

$$\text{total } d\bar{\sigma} \times \bar{A} = \left(dydz \frac{\partial A_z}{\partial x} dx\right)\hat{e}_z - \left(dydz \frac{\partial A_y}{\partial x} dx\right)\hat{e}_y$$

$$+ \left(dxdz \frac{\partial A_z}{\partial y} dy\right)\hat{e}_x - \left(dxdydz \frac{\partial A_x}{\partial y}\right)\hat{e}_z$$

$$- \left(dxdydz \frac{\partial A_y}{\partial z}\right)\hat{e}_x + \left(dxdydz \frac{\partial A_x}{\partial z}\right)\hat{e}_z$$

$$= dxdydz\left(\bar{\nabla} \times \bar{A}\right)$$

so $\oint_S d\bar{\sigma} \times A = \int_V d\tau (\bar{\nabla} \times \bar{A})$ as $S \to 0$

as $V \to 0$, $\int_V d\tau (\bar{\nabla} \times \bar{A}) \to (\bar{\nabla} \times \bar{A}) \int_V d\tau$

so $\frac{\oint_S d\bar{\sigma} \times \bar{A}}{\int_V d\tau} \to \bar{\nabla} \times \bar{A} \frac{\int_V d\tau}{\int_V d\tau} = \bar{\nabla} \times \bar{A}$ QED

4

HW 3.1. $\bar{V} = (1,1,1)$

at point $(1,2,1)$, what are basis vectors of a cylindrical/spherical system?

a.) cylindrical: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$ $\Big\}$ $F = (r\cos(\theta), r\sin(\theta), z)$ θ = angle often called "φ"

so $\hat{q}_r = \frac{1}{h_r}\frac{\partial \bar{F}}{\partial r} = \frac{1}{h_r}(\cos(\theta), \sin(\theta), 0) = (\cos(\theta), \sin(\theta), 0)$ $h_r = 1$

$\hat{q}_\theta = \frac{1}{h_\theta}\frac{\partial \bar{F}}{\partial \theta} = \frac{1}{h_\theta}(-r\sin(\theta), r\cos(\theta), 0) = (-\sin(\theta), \cos(\theta), 0)$ $h_\theta = r$

$\hat{q}_z = \frac{1}{h_z}\frac{\partial \bar{F}}{\partial z} = \frac{1}{h_z}(0,0,1) = (0,0,1)$ $h_z = 1$

can examine matrix form in order to express $(1,1,1)$ in terms of the cylindrical basis.

$$\begin{bmatrix}\hat{q}_r\\ \hat{q}_\theta\\ \hat{q}_z\end{bmatrix} = [t]\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix} \qquad t_{ij} = \hat{e}_j \cdot \hat{q}_i = \frac{1}{h_i}\frac{\partial \bar{F}}{\partial q_i}\cdot\hat{e}_j = \frac{1}{h_i}\frac{\partial x_j}{\partial q_i}$$

$$\begin{bmatrix}\hat{q}_r\\ \hat{q}_\theta\\ \hat{q}_z\end{bmatrix} = \begin{bmatrix}\cos(\theta) & \sin(\theta) & 0\\ -\sin\theta & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix}$$

, which indeed conforms to the relationships established above.

since $[t]^{-1} = [t]^T$,

$$\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix} = \begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{q}_r\\ \hat{q}_\theta\\ \hat{q}_z\end{bmatrix}$$

so $\begin{bmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{bmatrix} = \begin{bmatrix}\cos(\theta)\hat{q}_r - \sin(\theta)\hat{q}_\theta\\ \sin(\theta)\hat{q}_r + \cos(\theta)\hat{q}_\theta\\ \hat{q}_z\end{bmatrix}$ or $\hat{e}_1 + \hat{e}_2 + \hat{e}_3 = (\cos(\theta)+\sin(\theta))\hat{q}_r + (\cos(\theta)-\sin(\theta))\hat{q}_\theta + \hat{q}_z$

part a): $\theta = \tan^{-1}(2)$, $r = \sqrt{5}$ $\cos(\theta) = \frac{1}{\sqrt{5}}$; $\sin(\theta) = \frac{2}{\sqrt{5}} \rightarrow \bar{V} = \frac{3}{\sqrt{5}}\hat{q}_r - \frac{1}{\sqrt{5}}\hat{q}_\theta + \hat{q}_z$

part c): $\theta = \tan^{-1}(\frac{0}{0})$, $r = 1$ $\rightarrow$ undefined

4, cont'd

spherical: $x = r\sin(\theta)\cos(\varphi)$, $y = r\sin(\theta)\sin(\varphi)$, $z = r\cos(\theta)$ $\Big\}$ $\bar{F} = (r\sin(\theta)\cos(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\theta))$

θ, φ conventional

so:

$$\hat{q}_r = \frac{1}{h_r}\frac{\partial \bar{F}}{\partial r} = \frac{1}{h_r}\left(\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)\right)$$

$$h_r = 1$$

$$\therefore \boxed{\hat{q}_r = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))}$$

(Cartesian components of $\hat{q}_r$)

$$q_\theta = \frac{1}{h_\theta}\frac{\partial \bar{F}}{\partial \theta} = \frac{1}{h_\theta}\left(r\cos(\theta)\cos(\varphi), r\cos(\theta)\sin(\varphi), -r\sin(\theta)\right)$$

$$h_\theta = r$$

$$\therefore \boxed{\hat{q}_\theta = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), -\sin(\theta))}$$

(Cartesian components of $\hat{q}_\theta$)

$$\hat{q}_\varphi = \frac{1}{h_\varphi}\left(-r\sin(\theta)\sin(\varphi), r\sin(\theta)\cos(\varphi), 0\right) \rightarrow h_\varphi = r\sin(\theta)$$

$$\therefore \boxed{\hat{q}_\varphi = (-\sin(\varphi), \cos(\varphi), 0)}$$

(Cartesian components of $\hat{q}_\varphi$)

use these to directly write matrix:

$$\begin{bmatrix} \hat{q}_r \\ \hat{q}_\theta \\ \hat{q}_\varphi \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\varphi) & \sin(\theta)\sin(\varphi) & \cos(\theta) \\ \cos(\theta)\cos(\varphi) & \cos(\theta)\sin(\varphi) & -\sin(\theta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

so
$$\begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\varphi) & \cos(\theta)\cos(\varphi) & -\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & \cos(\theta)\sin(\varphi) & \cos(\varphi) \\ \cos(\theta) & -\sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} \hat{q}_r \\ \hat{q}_\theta \\ \hat{q}_\varphi \end{bmatrix} \rightarrow \hat{e}_x + \hat{e}_y + \hat{e}_z =$$

$$(\sin(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi) + \cos(\theta))\hat{q}_r$$
$$+ (\cos(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi) - \sin(\theta))\hat{q}_\theta$$
$$+ (-\sin(\varphi) + \cos(\varphi))\hat{q}_\varphi$$

part b) $\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$; $\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(2) \rightarrow \bar{V} = \frac{4}{\sqrt{6}}\hat{q}_r - \frac{2}{\sqrt{30}}\hat{q}_\theta - \frac{1}{\sqrt{5}}\hat{q}_\varphi$

part d) $\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}(1)$; $\varphi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{0}\right) \rightarrow$ undefined.

5 (KW 3.3)

a) Cartesian: position is $\sqrt{2}(\cos(\theta), \sin(\theta))$, at any θ; here, $\theta = 45°$.

velocity is $\sqrt{2}(-\sin(\theta), \cos(\theta))\frac{d\theta}{dt}$

$$= \sqrt{2}\omega_0(-\sin(\theta), \cos(\theta)) = \sqrt{2}\omega_0\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \omega_0(-1, 1)$$

b) $\left.\begin{array}{l} x = r\cos(\theta) \\ y = r\sin(\theta) \end{array}\right\}, \bar{r} = (r\cos(\theta), r\sin(\theta))$

$$\hat{q}_\theta = \frac{1}{h_\theta}\frac{\partial \bar{r}}{\partial \theta} = \frac{1}{h_\theta}(-r\sin(\theta), r\cos(\theta)) \rightarrow h_\theta = r$$

$$\boxed{\hat{q}_\theta = (-\sin(\theta), \cos(\theta))}$$

$$\hat{q}_r = \frac{1}{h_r}\frac{\partial \bar{r}}{\partial r} = \frac{1}{h_r}(\cos(\theta), \sin(\theta)) \rightarrow h_r = 1 \rightarrow \boxed{\hat{q}_r = (\cos(\theta), \sin(\theta))}$$

$$\begin{bmatrix} \hat{q}_r \\ \hat{q}_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \hat{q}_r \\ \hat{q}_\theta \end{bmatrix}$$

so $\omega_0(-\hat{e}_1 + \hat{e}_2) = \omega_0(-\cos(\theta)\hat{q}_r + \sin(\theta)\hat{q}_\theta + \sin(\theta)\hat{q}_r + \cos(\theta)\hat{q}_\theta)$

$$= \omega_0\left((\sin(\theta) - \cos(\theta))\hat{q}_r + (\sin(\theta) + \cos(\theta))\hat{q}_\theta\right)$$

however, by using $\omega_0(-\hat{e}_1 + \hat{e}_2)$, we are already assuming the given point.

so the answer is $\omega_0\left(\frac{2}{\sqrt{2}}\right)\hat{q}_\theta = \sqrt{2}\omega_0\hat{q}_\theta$.

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could also say that the polar position is $\sqrt{2}\hat{q}_r + 0\hat{q}_\theta$

$$\frac{d}{dt}\hat{q}_r = (-\sin(\theta)\hat{e}_1 + \cos(\theta)\hat{e}_2)\dot{\theta}$$

$$= \omega(-\sin(\theta), \cos(\theta)) = \omega\hat{q}_\theta$$

$$\left(\frac{d}{dt}\hat{q}_\theta = (-\cos(\theta), -\sin(\theta))\dot{\theta} = -\omega\hat{q}_r\right)$$

so velocity is $\sqrt{2}\omega\hat{q}_\theta$

6 (HW 3.5).

$\vec{r} = (x, y, z)$

$d\vec{r} = (dx, dy, dz)$

a) Cartesian.

what is $\hat{e}_\theta$? it is $\hat{e}_\theta$ defined in spherical, as

$\cos(\theta)\cos(\varphi)\,\hat{e}_x + \cos(\theta)\sin(\varphi)\,\hat{e}_y - \sin(\theta)\,\hat{e}_z$ $\qquad$ $\varphi = 0$ in x-z plane

so $\boxed{\hat{e}_\theta = \cos(\theta)\,\hat{e}_x - \sin(\theta)\,\hat{e}_z}$

on the line, everything can be parametrized in terms of x:

$\vec{r} = (x, 0, \sqrt{1-x^2})$ $\qquad$ $0 < x < 1$

$d\vec{r} = (dx, 0\ \frac{-x}{\sqrt{1-x^2}}dx)$. note that $x = \sin(\theta)$, so $\hat{e}_\theta = (\sqrt{1-x^2}, 0, -x)$

$$\hat{e}_\theta \cdot d\vec{r} = \left(\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}\right)dx$$

$$\int_1^0 \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_1^0 \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x)\Big|_1^0 = -\frac{\pi}{2}$$

b) cylindrical. $\hat{e}_\theta = \cos(\theta)\,\hat{e}_x - \sin(\theta)\,\hat{e}_z$.

in cylindrical (see #4), $\hat{e}_x = \cos(\varphi)\,\hat{q}_r - \sin(\varphi)\,\hat{q}_\phi = \hat{q}_r$

$\hat{e}_z = \hat{q}_z$

so $\boxed{\hat{e}_\theta = \cos(\theta)\,\hat{q}_r - \sin(\theta)\,\hat{q}_z}$

$\vec{r} = r\hat{q}_r + \sqrt{1-r^2}\,\hat{q}_z$ $\qquad$ $0 < r < 1$

$d\vec{r} = dr\,\hat{q}_r + \frac{-r\,dr}{\sqrt{1-r^2}}\hat{q}_z$ $\qquad$ $r = \sin(\theta)$

$\boxed{\hat{e}_\theta = \sqrt{1-r^2}\,\hat{q}_r - r\hat{q}_z}$

$d\vec{r} \cdot \hat{e}_\theta = \left(\sqrt{1-r^2} + \frac{r^2}{\sqrt{1-r^2}}\right)dr$

$\int d\vec{r} \cdot \hat{e}_\theta = \int_1^0 \left(\sqrt{1-r^2} + \frac{r^2}{\sqrt{1-r^2}}\right)dr = -\frac{\pi}{2}$, as above.

c) Spherical: $\hat{q}_\theta = \hat{e}_\theta$

$$\bar{r} = 1\,\hat{q}_r + 0\,q_\varphi + \theta\,\hat{e}_\theta$$

$$d\bar{r} = d\theta\,\hat{q}_\theta$$

$$\int \hat{e}_\theta \cdot d\bar{r} = \int_{\frac{\pi}{2}}^{0} d\theta = -\frac{\pi}{2}$$

7. a) $\bar{B} = B_0 \hat{e}_z = \bar{\nabla}\times\bar{A}$

$$\left.\begin{aligned} \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} &= B_0 \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} &= 0 \\ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} &= 0 \end{aligned}\right\} \text{ satisfied by } \bar{A} = (0, B_0 x, 0)$$

b) $\nabla\times B = 0$, so $B = \nabla\Phi$

$\Phi = B_0 z$ satisfies $\nabla\Phi = B_0 \hat{e}_z$

c) $\bar{B} = \frac{B_0}{r_0} r \hat{e}_\varphi = \nabla\times A = (\nabla\times A)_\varphi = \boxed{\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{B_0}{r_0} r}$

all other components of $\nabla\times A$ are 0.

$$\frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \frac{\partial A_\varphi}{\partial z}$$

$$\frac{\partial (r A_\varphi)}{\partial r} = \frac{\partial A_r}{\partial \varphi}$$

can satisfy by setting $A_r = A_\varphi = 0$

$A_z = -\frac{B_0}{2r_0} r^2$

$\bar{A} = \left(0, 0, -\frac{B_0}{2r_0} r^2\right)$

d) $\bar{\nabla}\times\bar{B} = -\frac{\partial B_\varphi}{\partial z} a_r + \frac{1}{r}\frac{\partial}{\partial r}(r B_\varphi) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{B_0}{r_0} r^2\right) = \frac{2B_0 r}{r_0} \cdot \frac{1}{r} = \boxed{\frac{2B_0}{r_0}}$; $\neq 0$

$\bar{\nabla}\times\bar{B} \neq 0 \rightarrow \bar{B}$ can't be written as $\nabla\Phi$