① (KW 4.4) : $u = x^2 - y^2$, $v = 2xy$, $z = z$ $\longrightarrow$ $u^2 + v^2 = (x^2+y^2)^2$

$$a_{ij} = \frac{h_i' \partial q_i'}{h_j \partial q_j}$$ [curvilinear → curvilinear]

In this case, going from Cartesian → curvilinear => all Cartesian scale factors = 1

$$a_{ij} = \frac{h_i' \partial q_i'}{\partial q_j}$$

$$[a] = \begin{bmatrix} h_u \frac{\partial u}{\partial x} & h_u \frac{\partial u}{\partial y} & h_u \frac{\partial u}{\partial z} \\ h_v \frac{\partial v}{\partial x} & h_v \frac{\partial v}{\partial y} & h_v \frac{\partial v}{\partial z} \\ h_z \frac{\partial z}{\partial x} & h_z \frac{\partial z}{\partial y} & h_z \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} h_u(2x) & h_u(-2y) & 0 \\ h_v(2y) & h_v(2x) & 0 \\ 0 & 0 & h_z \end{bmatrix}$$

$$\hat{q}_i' = a_{ij} \hat{e}_j$$

$$\begin{bmatrix} \hat{q}_u \\ \hat{q}_v \\ \hat{q}_z \end{bmatrix} = \begin{bmatrix} h_u \frac{\partial u}{\partial x} & h_u \frac{\partial u}{\partial y} & h_u \frac{\partial u}{\partial z} \\ h_v \frac{\partial v}{\partial x} & h_v \frac{\partial v}{\partial y} & h_v \frac{\partial v}{\partial z} \\ h_z \frac{\partial z}{\partial x} & h_z \frac{\partial z}{\partial y} & h_z \frac{\partial z}{\partial z} \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

$$\hat{q}_u = h_u\left(\frac{\partial u}{\partial x}\hat{e}_x + \frac{\partial u}{\partial y}\hat{e}_y + \frac{\partial u}{\partial z}\hat{e}_z\right) = h_u\left[(2x)\hat{e}_x + (-2y)\hat{e}_y\right]$$

$$\hat{q}_v = h_v\left(\frac{\partial v}{\partial x}\hat{e}_x + \frac{\partial v}{\partial y}\hat{e}_y + \frac{\partial v}{\partial z}\hat{e}_z\right) = h_v\left[(2y)\hat{e}_x + (2x)\hat{e}_y\right]$$

$$\hat{q}_z = h_z\left(\frac{\partial z}{\partial x}\hat{e}_x + \frac{\partial z}{\partial y}\hat{e}_y + \frac{\partial z}{\partial z}\hat{e}_z\right) = h_z \hat{e}_z$$

But we know that $\hat{q}_u, \hat{q}_v, \hat{q}_z$ are unit vectors, so their magnitudes must be 1

$$|\hat{q}_u|^2 = h_u^2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = h_u^2(4x^2+4y^2) = 1 \implies h_u = \tfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}} = \tfrac{1}{2}(u^2+v^2)^{-\frac{1}{4}}$$

$$|\hat{q}_v|^2 = h_v^2\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] = h_v^2(4y^2+4x^2) = 1 \implies h_v = \tfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}} = \tfrac{1}{2}(u^2+v^2)^{-\frac{1}{4}}$$

$$|\hat{q}_z|^2 = h_z^2 = 1 \implies h_z = 1$$

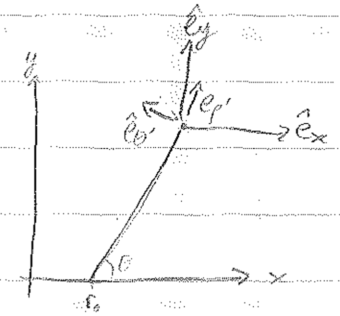
$$[a] = \begin{bmatrix} x(x^2+y^2)^{-\frac{1}{2}} & -y(x^2+y^2)^{-\frac{1}{2}} & 0 \\ y(x^2+y^2)^{-\frac{1}{2}} & x(x^2+y^2)^{-\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} r_u \\ r_v \\ r_z \end{bmatrix} = \begin{bmatrix} h_u(2x) & h_u(-2y) & 0 \\ h_v(2y) & h_v(2x) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2h_u(x^2-y^2) \\ 2h_v(2xy) \\ z \end{bmatrix} = \begin{bmatrix} 2h_u \cdot u \\ 2h_v \cdot v \\ z \end{bmatrix} = \begin{bmatrix} (u^2+v^2)^{-\frac{1}{4}} \cdot u \\ (u^2+v^2)^{-\frac{1}{4}} \cdot v \\ z \end{bmatrix}$$

$$\vec{r} = u(u^2+v^2)^{-\frac{1}{4}}\hat{q}_u + v(u^2+v^2)^{-\frac{1}{4}}\hat{q}_v + z\hat{q}_z$$

2. (KW 4.7)

a)



b) $x = r_0 + \rho' \cos(\theta)$

$y = \rho' \sin(\theta)$

$\rho = \sqrt{(x-r_0)^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x-r_0}\right)$

$F = (x, y)$

c) $h_x = h_y = 1$

$h_{\rho'} = \sqrt{\left(\frac{\partial x}{\partial \rho'}\right)^2 + \left(\frac{\partial x}{\partial \theta'}\right)^2} = 1$

$h_{\theta'} = \sqrt{\left(\frac{\partial x}{\partial \theta'}\right)^2 + \left(\frac{\partial y}{\partial \theta'}\right)^2} = \rho'$

$t_{ij} = \frac{h_j}{h_i''} \frac{\partial q_j}{\partial q_i''}$

where $q_j = \rho'; \theta'$

$q_i'' = x, y$

(we will eventually transpose)

so $t_{x\rho'} = \frac{h_{\rho'}}{h_x} \frac{\partial \rho'}{\partial x} = h_{\rho'} \frac{\partial \rho'}{\partial x} = \frac{x-r_0}{\sqrt{(x-r_0)^2+y^2}}$

$t_{x\theta'} = \frac{h_{\theta'}}{h_x} \frac{\partial \theta'}{\partial x} = \rho' \cdot \frac{1}{1+\left(\frac{y}{x-r_0}\right)^2} = \frac{\sqrt{(x-r_0)^2+y^2}}{1+\left(\frac{y}{x-r_0}\right)^2} \cdot \left(\frac{-y}{(x-r_0)^2}\right) = \frac{-y}{\sqrt{(x-r_0)^2+y^2}}$

$t_{y\theta'} = \sqrt{(x-r_0)^2+y^2} \cdot \frac{1}{1+\left(\frac{y}{x-r_0}\right)^2} \cdot \frac{1}{x-r_0} = \sqrt{(x-r_0)^2+y^2} \cdot \frac{x-r_0}{(x-r_0)^2+y^2} = \frac{x-r_0}{\sqrt{(x-r_0)^2+y^2}}$

$t_{y\rho'} = \frac{y}{\sqrt{(x-r_0)^2+y^2}}$

$[a]^T = \frac{1}{\sqrt{(x-r_0)^2+y^2}} \begin{bmatrix} x-r_0 & y \\ -y & x-r_0 \end{bmatrix} \Rightarrow [a] = \frac{1}{\sqrt{(x-r_0)+y^2}} \begin{bmatrix} x-r_0 & -y \\ y & x-r_0 \end{bmatrix}$

d) $d\bar{r} = dx\,\hat{e}_x + dy\,\hat{e}_y = d\rho'\,\hat{e}_{\rho'} + \rho d\theta'\,\hat{e}_{\theta'}$

3. (KW 4.12)

$\bar{J} = \bar{\bar{\sigma}} \cdot \bar{E}$ $\sigma = 1\,\hat{e}_x\hat{e}_x + 2\,\hat{e}_y\hat{e}_y$

a) $E = E_0\hat{e}_x + E_0\hat{e}_y$

$\bar{\bar{\sigma}} \cdot \bar{E} = E_0\hat{e}_x + 2E_0\hat{e}_y$

b) $\hat{e}_x' = \frac{1}{\sqrt{2}}\hat{e}_x + \frac{1}{\sqrt{2}}\hat{e}_y$ $x' = x + y$

$\hat{e}_y' = -\frac{1}{\sqrt{2}}\hat{e}_x + \frac{1}{\sqrt{2}}\hat{e}_y$ $y' = -x + y$

both systems are cartesian → all $h = 1$ →

$dx' = a_y dx$

$$\begin{bmatrix} \hat{e}_x' \\ \hat{e}_y' \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \end{bmatrix}$$

$$\bar{\bar{\sigma}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

so $\bar{\bar{\sigma}} = \frac{3}{2}\hat{e}_x'\hat{e}_x' + \frac{3}{2}\hat{e}_y'\hat{e}_y' + \frac{1}{2}\hat{e}_x'\hat{e}_y' + \frac{1}{2}\hat{e}_y'\hat{e}_x'$

$\bar{\bar{\sigma}} \cdot \bar{E} = \frac{3}{2}\hat{e}_x'\hat{e}_x' \cdot \sqrt{2}E_0\hat{e}_x' + \frac{1}{2}\hat{e}_y'\hat{e}_x' \cdot \sqrt{2}E_0\hat{e}_x'$ (non-vanishing terms)

$= \frac{3}{\sqrt{2}}E_0\hat{e}_x' + \frac{1}{\sqrt{2}}E_0\hat{e}_y'$

$= \frac{3}{\sqrt{2}}E_0\left(\frac{1}{\sqrt{2}}\hat{e}_x + \frac{1}{\sqrt{2}}\hat{e}_y\right) + \frac{1}{\sqrt{2}}E_0\left(\frac{-1}{\sqrt{2}}\hat{e}_x + \frac{1}{\sqrt{2}}\hat{e}_y\right)$

$= E_0\hat{e}_x + 2E_0\hat{e}_y$ ✓

4. (HW 4.15)

a) iv) $(\bar{V}\cdot\bar{\bar{T}})\times(\bar{\bar{T}}\cdot\bar{V})$

$= \left[(V_i\hat{e}_i\cdot T_{jk}\hat{e}_j\hat{e}_k)\cdot e_l\right]\epsilon_{lmn}\left[(T_{ab}\hat{e}_a\hat{e}_b\cdot V_c\hat{e}_c)\cdot\hat{e}_m\right]\hat{e}_n$

$= \left[V_i T_{ik}\hat{e}_k\cdot\hat{e}_l\right]\epsilon_{lmn}\left[T_{ab}V_b\hat{e}_a\cdot\hat{e}_m\right]\hat{e}_n$

$= \left[V_i T_{il}\right]\left[T_{mb}V_b\right]\epsilon_{lmn}\hat{e}_n$

b) $\bar{V}\times(\bar{V}\cdot\bar{\bar{T}}) = \bar{V}\times(V_i\hat{e}_i\cdot T_{jk}\hat{e}_j\hat{e}_k)$

$= \bar{V}\times(V_j T_{jk}\hat{e}_k)$

$= \epsilon_{abc}V_a V_j T_{jb}\hat{e}_c$ for *any* V.

$= 0$

$\epsilon_{ab}\underbrace{V_a V_j T_{jb}}_{} = 0$

must be symmetric in a, b

$V_a T_{jb} = V_b T_{ja} \rightarrow T_{ja} = \frac{V_a}{V_b}T_{jb}$

5 (HW 5.17)

a) $a_{xu} = \frac{h_u}{h_x}\frac{\partial u}{\partial x}$ $\qquad h_x = h_y = 1.$

$a_{xv} = \frac{h_v}{h_x}\frac{\partial v}{\partial x}$ $\qquad h_u = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}$

$\qquad h_v = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2}$

$a_{yu} = \frac{h_u}{h_y}\frac{\partial u}{\partial y}$

$a_{yv} = \frac{h_v}{h_y}\frac{\partial v}{\partial y}$ $\qquad u = \frac{x}{x^2+y^2}$; $v = \frac{-y}{x^2+y^2}$

$$1 = \frac{(x^2+y^2)\frac{dx}{du} - x\cdot\left(2x\frac{dx}{du} + 2y\frac{dy}{du}\right)}{(x^2+y^2)^2}$$

$$0 = \frac{(x^2+y^2)\left(-\frac{dy}{du}\right) + y\left(2x\frac{dx}{du} + 2y\frac{dy}{du}\right)}{(x^2+y^2)^2}$$

$$0 = \frac{(x^2+y^2)\frac{dx}{dv} - x\left(2x\frac{dx}{dv} + 2y\frac{dy}{dv}\right)}{(x^2+y^2)^2}$$

$$1 = \frac{-(x^2+y^2)\frac{dy}{dv} + y\left(2x\frac{dx}{dv} + 2y\frac{dy}{dv}\right)}{(x^2+y^2)^2}$$

$$\begin{bmatrix} y^2-x^2 & -2yx \\ 2xy & y^2-x^2 \end{bmatrix}\begin{bmatrix} \frac{dx}{dv} \\ \frac{dy}{dv} \end{bmatrix} = \begin{bmatrix} 0 \\ (x^2+y^2)^2 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{dx}{dv} \\ \frac{dy}{dv} \end{bmatrix} = \begin{bmatrix} y^2-x^2 \\ 2xy \end{bmatrix}$$

$$\begin{bmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2yx}{(x^2+y^2)^2} \\ 2xy & y^2-x^2 \end{bmatrix}\begin{bmatrix} \frac{dx}{du} \\ \frac{dy}{du} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} y^2-x^2 \\ -2xy \end{bmatrix} = \begin{bmatrix} \frac{dx}{du} \\ \frac{dy}{du} \end{bmatrix}$$

$$h_v = h_u = \sqrt{(y^2-x^2)+(2xy)^2} = \sqrt{(x^2+y^2)^2} = x^2+y^2$$

$t_{ij} = \frac{h_{j''}}{h_i}\frac{\partial q_i}{\partial q_i''}$ if going ~~from~~ cartesian, q_i'' are x, y, z

$$[t]^T = \begin{bmatrix} \frac{\partial x}{h_u\partial u} & \frac{\partial y}{h_u\partial u} & \frac{\partial z}{h_u\partial u} \\ \frac{\partial x}{h_v\partial v} & \frac{\partial y}{h_v\partial v} & \frac{\partial z}{h_v\partial v} \\ \frac{\partial x}{h_z\partial z} & \frac{\partial y}{h_z\partial z} & \frac{\partial z}{h_z\partial z} \end{bmatrix} = \begin{bmatrix} \frac{y^2-x^2}{y^2+x^2} & \frac{-2xy}{x^2+y^2} & 0 \\ \frac{2xy}{x^2+y^2} & \frac{y^2-x^2}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5.

$$\begin{bmatrix} \hat{q}_u \\ \hat{q}_v \\ \hat{q}_z \end{bmatrix} = [t] \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

at $x=1, y=1, z=1,$ $[t] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b)

$$\begin{bmatrix} \bar{V}\cdot\hat{q}_u \\ \bar{V}\cdot\hat{q}_v \\ \bar{V}\cdot\hat{q}_z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \bar{V} = -4\hat{q}_u - \cancel{2}\hat{q}_v + 3\hat{q}_z$$

c)

$$[T'] = [t]'[T][t]^T = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 3 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -3 \\ 0 & 0 & 1 \\ -4 & 0 & 0 \end{bmatrix}$$

6. (KW 4.18)

$$\bar{\bar{I}} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

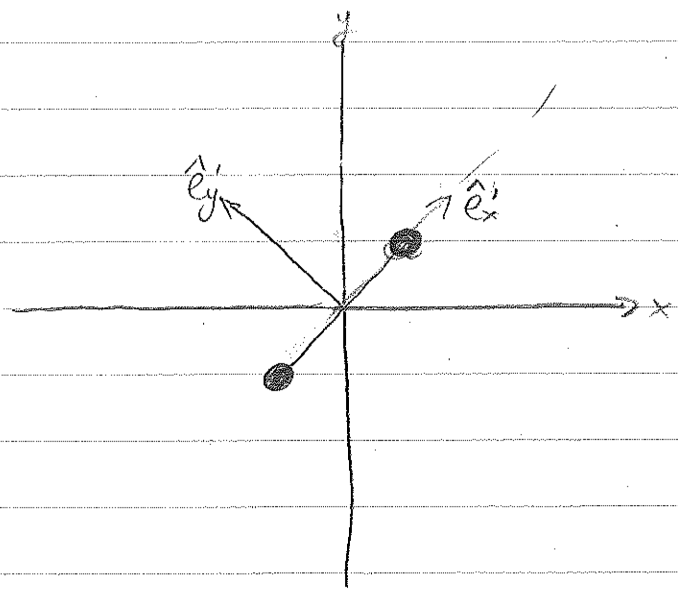
eigenvalues and eigenvectors.

λ	$\hat{e}_i'$
4	$\rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \hat{e}_z'$
4	$\rightarrow -\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \hat{e}_y'$
0	$\rightarrow \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \hat{e}_x'$

to maintain handedness

check: $$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \checkmark$$



7. (KW 4.21)

$\bar{\bar{T}} = T_{ij}\,\hat{e}_i\hat{e}_j$

$T_{ij}\,a_{nj} = \lambda_n a_{ni}$ (not summing over n)

$T^*_{ij}\,a^*_{nj} = \lambda^*_n a^*_{ni}$

$a^*_{mi}\,T_{ij}\,a_{nj} = a^*_{mi}\,\lambda_n a_{ni}$

$a^*_{ni}\,T_{ij}\,a_{mj} = a^*_{ni}\,\lambda_m a_{mi}$

difference $= a^*_{mi}\lambda_n a_{ni} - a^*_{ni}\lambda_m a_{mi} = a^*_{mi}T_{ij}a_{nj} - a^*_{ni}T_{ij}a_{mj}$

$= a^*_{mj}\lambda^*_m a_{nj} - a^*_{nj}\lambda^*_n a_{mj}$

so $a^*_{mi}\lambda_n a_{ni} + a^*_{ni}\lambda^*_n a_{mi} = a^*_{nj}\lambda^*_m a_{nj} + a^*_{nj}\lambda_m a_{mj}$

since i, j are dummies

so $\boxed{2\mathrm{Re}[a_{mi}\lambda^*_n a^*_{ni}] = 2\mathrm{Re}[a^*_{ni}\lambda_m a_{mi}]}$ (1)

likewise by adding rather than subtracting,

$a^*_{mi}\lambda_n a_{ni} + a^*_{ni}\lambda_m a_{mi} = a^*_{mi}\lambda^*_m a_{ni} + a^*_{ni}\lambda^*_n a_{mi}$

$\mathrm{Im}[a^*_{mi}\lambda_n a_{ni}] = \mathrm{Im}[a^*_{ni}\lambda^*_m a_{mi}]$

so $\lambda_n a_{ni} a^*_{mi} = \lambda_m a^*_{ni} a_{mi}$

let $n = m$.

$\boxed{\mathrm{Im}[\underbrace{a^*_{mi}\,a_{mi}}_{real}\lambda_m] = \mathrm{Im}[\underbrace{a^*_{mi}\,a_{mi}}_{real}\lambda^*_m]}$ (2)

$\mathrm{Im}[\lambda_m] = \mathrm{Im}[\lambda^*_m] \Rightarrow \lambda_m$ is real.

say $n \neq m$

$\lambda_n \mathrm{Im}[a^*_{mi}\,a_{ni}] = \lambda_m \mathrm{Im}[a^*_{ni}\,a_{mi}]$ since we've established λ real.

assuming no degeneracy, $\lambda_n \neq \lambda_m$

so $\underbrace{a^*_{mi}\,a_{ni}}_{\text{dot product}}$ must be real.

but by (1), assuming no degeneracy,

$\mathrm{Re}[a_{mi}\,a^*_{ni}] = a_{mi}a^*_{ni} = 0 \Rightarrow$ orthogonality