ALP 4210
HW #8 Solutions

1. (KW 6.2)

i) $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$ say $z = x+iy$ $x, y \in \mathbb{R}$

$\frac{1}{2i}(e^{iz} - e^{-iz}) = 0$

$e^{ix}e^{-y} - e^{-ix}e^{y} = 0$

$e^{y-ix} = e^{-(y-ix)}$ only holds if $y=0$, $ix = -ix + 2\pi n$ n integer

$x = 2\pi n - x \rightarrow \boxed{x = \pi n}$

$\boxed{z = n\pi}$

ii) $\frac{1}{2}(e^{iz} + e^{-iz}) = 0$

$e^{iz} = -e^{-iz}$

$e^{2iz} = -1 \rightarrow 2z = (2n+1)\pi \rightarrow \boxed{z = (n+\frac{1}{2})\pi}$

iii) $\frac{1}{2}(e^{z} - e^{-z}) = 0$

$e^{z} = e^{-z} \rightarrow e^{2z} = 1 \rightarrow 2z = 2\pi n i \rightarrow \boxed{z = n\pi i}$

iv) $\frac{1}{2}(e^{z} + e^{-z}) = 0 \rightarrow e^{z} = -e^{-z} \rightarrow e^{2z} = -1 \rightarrow 2z = (2n+1)\pi i \rightarrow \boxed{z = (n+\frac{1}{2})\pi i}$

2. (KW 6.6.) $\boxed{z = x + iy}$ $x = Re(z)$; $y = Im(z)$

i) $e^{z} = e^{x+iy} = e^{x} e^{iy} = e^{x}(\cos(y) + i\sin(y))$

$Re[e^{z}] = e^{x}\cos(y)$

$Im[e^{z}] = e^{x}\sin(y)$

v) $\frac{1}{z^2} = \frac{z^* z^*}{z z z^* z^*} = \frac{(z^*)^2}{|z|^4} = \frac{(x-iy)(x-iy)}{(x^2+y^2)^2} = \frac{x^2 - y^2 - 2iyx}{(x^2+y^2)^2}$

$Re\left[\frac{1}{z^2}\right] = \frac{x^2 - y^2}{(x^2+y^2)^2}$

$Im\left[\frac{1}{z^2}\right] = \frac{-2yx}{(x^2+y^2)^2}$

analyticity:

i) $u = e^{x}\cos y$ $\quad \frac{\partial u}{\partial x} = u \quad \frac{\partial u}{\partial y} = -v$

$v = e^{x}\sin(y)$ $\quad \frac{\partial v}{\partial x} = v \quad \frac{\partial v}{\partial y} = u$

which obviously satisfy C-R relations everywhere $\rightarrow$ analytic everywhere

v) $u = \frac{x^2 - y^2}{(x^2+y^2)^2}$ $\quad \frac{\partial u}{\partial x} = \frac{-2x(x^2 - 3y^2)}{(x^2+y^2)^3}$ $\quad \frac{\partial v}{\partial y} = \frac{-2x(x^2 - 3y^2)}{(x^2+y^2)^3}$

$v = \frac{-2yx}{(x^2+y^2)^2}$ $\quad \frac{\partial u}{\partial y} = \frac{2y(y^2 - 3x^2)}{(x^2+y^2)^3}$ $\quad \frac{\partial v}{\partial x} = \frac{-2y(y^2 - 3x^2)}{(x^2+y^2)^3}$

which also satisfy C-R relations everywhere

$\rightarrow$ analytic everywhere (undefined at origin)

3. (KW 6.6)

vi) $\ln(z^2+1) = \ln((x+iy)(x+iy)+1) = \ln(x^2-y^2+2ixy+1)$

$= \ln\left(\sqrt{(x^2+1-y^2)^2+(2xy)^2}\exp\left(i\tan^{-1}\left(\frac{2xy}{x^2-y^2+1}\right)\right)\right) = \ln\left(\sqrt{(x^2+1-y^2)^2+(2xy)^2}\right) + i\tan^{-1}\left(\frac{2xy}{1+x^2-y^2}\right)$

$Re(w) = \ln\left(\sqrt{(x^2+1-y^2)^2+(2xy)^2}\right) = u$

$Im(w) = \tan^{-1}\left(\frac{2xy}{x^2-y^2+1}\right) = v$

$\frac{\partial u}{\partial x} = \frac{2x(1+x^2+y^2)}{x^4+(y^2-1)^2+2x^2(y^2+1)} = \frac{\partial v}{\partial y}$ ✓

$\frac{\partial u}{\partial y} = \frac{2y(x^2+y^2-1)}{x^4+(y^2-1)^2+2x^2(1+y^2)} = -\frac{\partial v}{\partial x}$ ✓

so w is analytic everywhere. (Not defined at $\pm i$).

x) $\ln(z) = \ln(r) + i\theta \qquad 0 \leq \theta < 2\pi$ (which covers entire plane)

$z^* = re^{-i\theta}$

$w = re^{-i\theta}\ln(r) + i\theta re^{-i\theta}$

$= r\ln(r)(\cos\theta - i\sin\theta) + i\theta r(\cos\theta - i\sin\theta)$

$$\boxed{\begin{aligned} u &= r\ln(r)\cos\theta + r\theta\sin\theta = \ln(\sqrt{x^2+y^2})\,x + \tan^{-1}(\tfrac{y}{x})\,y \\ v &= -r\ln(r)\sin\theta + r\theta\cos\theta = -\ln(\sqrt{x^2+y^2})\,y + \tan^{-1}(\tfrac{y}{x})\,x \end{aligned}}$$

$\frac{\partial u}{\partial x} = \frac{x^2-y^2+\frac{1}{2}(x^2+y^2)\ln(x^2+y^2)}{x^2+y^2}$; $\frac{\partial v}{\partial y} = \frac{x^2-y^2-\frac{1}{2}(x^2+y^2)\ln(x^2+y^2)}{x^2+y^2} \Rightarrow (x^2+y^2)\ln(x^2+y^2) = 0$

$\Downarrow$

$\boxed{x^2+y^2=1}$

unit circle

$\frac{\partial u}{\partial y} = \frac{2xy}{x^2+y^2} + \tan^{-1}(\frac{y}{x})$; $\frac{\partial v}{\partial x} = \frac{-2xy}{x^2+y^2} + \tan^{-1}(\frac{y}{x})$

so, only analytic where $\tan^{-1}(\frac{y}{x}) = 0$

true wherever $y=0 \rightarrow$ only analytic for z real

however, unit circle only intersects real axis at 2 points, and a function cannot be analytic at isolated points $\rightarrow$ so its analytic nowhere.

4. (KW 6.7)

analyticity: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \rightarrow v = 3x^2y - y^3 + f(x)$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow -6xy = -\frac{\partial v}{\partial x} \rightarrow v = 3x^2y + f(y)$

$v = 3x^2y - y^3$ satisfies this

$w(z) = x^3 - 3xy^2 + i(3x^2y - y^3) = \underline{z^3}$

5. (KW 6.8)

We take the limit along constant Θ, then constant r, then equate
derivative $\quad z_0 = r_0 e^{i\theta_0}$;

constant θ: $$\frac{dw}{dz} = \lim_{\Delta r \to 0} \frac{w(r_0+\Delta r, \theta_0) - w(r_0, \theta_0)}{(r_0+\Delta r)e^{i\theta_0} - r_0 e^{i\theta_0}} = e^{-i\theta_0} \lim_{\Delta r \to 0} \frac{w(r_0+\Delta r, \theta_0) - w(r_0, \theta_0)}{\Delta r} \quad (1)$$

constant r: $$\frac{dw}{dz} = \lim_{\Delta\theta \to 0} \frac{w(r_0, \theta_0+\Delta\theta) - w(r_0, \theta_0)}{r_0 e^{i\theta_0}(e^{i\Delta\theta}-1)} = \frac{e^{-i\theta_0}}{r_0} \lim_{\Delta\theta \to 0} \frac{w(r_0, \theta_0+\Delta\theta) - w(r_0, \theta_0)}{e^{i\Delta\theta}-1} \quad (2)$$

$$(1) = \lim_{\Delta r \to 0} \frac{R(r_0+\Delta r, \theta_0) e^{i\Theta(r_0+\Delta r, \theta_0)} - R(r_0, \theta_0) e^{i\Theta(r_0, \theta_0)}}{\Delta r}$$ (omitting prefactor)

Taylor-expand to 1st order in Δr

$$= \lim_{\Delta r \to 0} \frac{\left(R(r_0, \theta_0) + \frac{\partial R}{\partial r}\Big|_{\theta_0, r_0} \Delta r\right) e^{i[\Theta(r_0, \theta_0)]} e^{i\left[\frac{\partial \Theta}{\partial r}\big|_{\theta_0, r_0} \Delta r\right]} - R(r_0, \theta_0) e^{i\Theta(r_0, \theta_0)}}{\Delta r}$$

let $R_0 \equiv R(r_0, \theta_0)$; $\Theta_0 = \Theta(r_0, \theta_0)$

$$= \lim_{\Delta r \to 0} \frac{1}{\Delta r}\left[R_0 e^{i\Theta_0}\left(e^{i\frac{\partial\Theta}{\partial r}\big|_{\theta_0, r_0}\Delta r}\right) + \left(\Delta r \frac{\partial R}{\partial r}\Big|_{\theta_0, r_0} e^{i\frac{\partial \Theta}{\partial r}\big|_{\theta_0, r_0}\Delta r} - R_0\right) e^{i\Theta_0}\right]$$

still need to expand exponentials, but omitting 2nd-order terms

$$= \lim_{\Delta r \to 0} \frac{1}{\Delta r}\left[R_0 e^{i\Theta_0}\left(1 + i\Delta r \frac{\partial\Theta}{\partial r}\Big|_{\theta_0, r_0}\right) + e^{i\Theta_0}\left[\frac{\partial R}{\partial r}\Big|_{\theta_0, r_0}\Delta r - R_0\right]\right]$$

$$= \lim_{\Delta r \to 0} \frac{e^{i\Theta_0}}{\Delta r}\left[i\Delta r R_0 \frac{\partial\Theta}{\partial r}\Big|_{\theta_0, r_0} + \frac{\partial R}{\partial r}\Big|_{\theta_0, r_0}\Delta r\right] = \boxed{e^{i\Theta_0}\left(\frac{\partial R}{\partial r}\Big|_{z_0} + iR_0 \frac{\partial\Theta}{\partial r}\Big|_{z_0}\right) = (}$$

what about (2)? again, omit prefactor of $e^{-i\theta_0}$ to equate to (1).

$$\frac{1}{r_0}\lim_{\Delta\theta \to 0} \frac{R(r_0, \theta_0+\Delta\theta) e^{i\Theta(r_0, \theta_0+\Delta\theta)} - R(r_0, \theta_0) e^{i\Theta(r_0, \theta_0)}}{e^{i\Delta\theta}-1}$$ expand to 1st order in $\Delta\theta$.

$$\frac{1}{r_0}\lim_{\Delta\theta \to 0} \frac{\left(R_0 + \frac{\partial R}{\partial \theta}\Big|_{z_0}\Delta\theta\right) e^{i\left(\Theta_0 + \frac{\partial\Theta}{\partial\theta}\big|_{z_0}\Delta\theta\right)} - R_0 e^{i\Theta_0}}{i\Delta\theta}$$

5 cont'd once again, expand exponentials to 1st order, and only keep 1st order terms.

$$\frac{1}{r_0} \lim_{\Delta\theta \to 0} \frac{\left(R_0 + \frac{\partial R}{\partial \theta}\Big|_{z_0} \Delta\theta\right) e^{i\Theta_0}\left(1 + i\frac{\partial \Theta}{\partial \theta}\Big|_{z_0} \Delta\theta\right) - R_0 e^{i\Theta_0}}{i\Delta\theta}$$

$$\frac{-ie^{i\Theta_0}}{r_0} \lim_{\Delta\theta \to 0} \frac{R_0 + iR_0 \frac{\partial \Theta}{\partial \theta}\Big|_{z_0} \Delta\theta + \frac{\partial R}{\partial \theta}\Big|_{z_0} \Delta\theta - R_0}{\Delta\theta}$$

$$= \frac{-ie^{i\Theta_0}}{r_0}\left(iR_0 \frac{\partial \Theta}{\partial \theta}\Big|_{z_0} + \frac{\partial R}{\partial \theta}\Big|_{z_0}\right)$$

so $\boxed{(2) = \frac{e^{i\Theta_0}}{r_0}\left(R_0 \frac{\partial \Theta}{\partial \theta}\Big|_{z_0} - i\frac{\partial R}{\partial \theta}\Big|_{z_0}\right)}$

- equating the real and imaginary parts of (1) and (2), we find

real: $\boxed{\frac{R_0}{r_0}\frac{\partial \Theta}{\partial \theta} = \frac{\partial R}{\partial r}}$

imaginary: $\boxed{R_0 \frac{\partial \Theta}{\partial r} = -\frac{1}{r_0}\frac{\partial R}{\partial \theta}}$ as required.

6. (KW 6.9)

$\oint_c (z-z_0)^n dz$ let $w = z - z_0$

$dw = dz$

$\oint_c w^n dw$ being a polynomial, w^n is analytic on all $\mathbb{C}$

so we can take our path along a circle of radius 1.

$w = e^{i\theta}$; $w^n = e^{in\theta}$; $dw = ie^{i\theta} d\theta$

$\int_0^{2\pi} ie^{i\theta} d\theta \, e^{in\theta} = i\int_0^{2\pi} e^{i(n+1)\theta} d\theta = \frac{i}{(n+1)i} e^{i(n+1)\theta}\Big|_0^{2\pi} = 0$ if $n \neq -1$

if $n = -1$,

$\int i e^{i\theta} d\theta \, e^{-i\theta} = i\int_0^{2\pi} d\theta = 2\pi i$

7. (KW 6.10)

ii) $\cos(z)$ expanded around $z=0$.

eq. 6.109 in book: $c_n = \frac{1}{n!}\frac{d^n f(z)}{dz^n}\Big|_{z=z_0}$, in this case $z_0 = 0$.

$$f'(z)\big|_{z_0} = -\sin(z) \xrightarrow{z=0} \emptyset$$

$$f''(z)\big|_{z_0} = -\cos(z_0) \to -1$$

etc.

so $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!}$, etc. $\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$

converges for any z, since cos is always well-defined.

$$\lim_{n\to\infty}\left|\frac{c_{n+2}\, z^{n+2}}{c_n\, z^n}\right| = \left|\frac{c_{n+2}\; z^2}{c_n}\right| = \left|\frac{n!}{(n+2)!} z^2\right| = \emptyset \text{ for any } z^2$$

iv) $\ln(1+z)$ expanded around $z_0 = 0$

$$f'\big|_{z_0} = \frac{1}{1+z_0} \to 1$$

$$f''\big|_{z_0} = \frac{-1}{(1+z_0)^2} \to -1$$

$$f'''\big|_{z_0} = -(-2)(1+z_0)^{-3} \to 2!$$

$$f''''\big|_{z_0} -(-2)(-3)(1+z_0)^{-4} \to -3!$$

etc.

so $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$

absolute convergence test: $\lim_{n\to\infty}\left|\frac{n}{n+1} z\right| = |z|$, so must converge for $|z|<1$.

- also, it doesn't converge for $z=-1$, so our disk of convergence is unit disk.

8. 16. a) $\left|\frac{z-z_0}{z-a}\right| < 1$ is condition for Taylor Series (6.123)

$a = -1$

$\left|\frac{z-z_0}{z+1}\right| < 1 \rightarrow |z-z_0| < |z+1|$ $z = 1+e^{i\theta}$, any point on the contour

$$\frac{d^n}{dz^n}\frac{1}{z+1} = \frac{(-1)^n n!}{(z_0+1)^{n+1}}$$

$$c_n = \frac{(-1)^n}{(z_0+1)^{n+1}}$$

b) (6.157) $|z-z_0| > |a-z_0|$ $a = -1$ $z = 1+e^{i\theta}$, any point on the contour.

$|z-z_0| > |z+1|$ z further from z_0 than from -1.

$$\frac{1}{z-a} = \sum_{n=-\infty}^{-1} \frac{-1}{(a-z_0)^{n+1}}(z-z_0)^n$$

as in class

$$c_n = (a-z_0)^{-(n+1)}$$ $a = -1$

$$= (-1-z_0)^{-n-1} = \frac{1}{(-1)^{n+1}(1+z_0)^{n+1}}$$

$$= \left(\frac{-1}{1+z_0}\right)^{n+1}$$