

Flag Enumeration in Polytopes, Eulerian Partially Ordered Sets and Coxeter Groups

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Abstract. We discuss the enumeration theory for flags in Eulerian partially ordered sets, emphasizing the two main geometric and algebraic examples, face posets of convex polytopes and regular CW -spheres, and Bruhat intervals in Coxeter groups. We review the two algebraic approaches to flag enumeration – one essentially as a quotient of the algebra of noncommutative symmetric functions and the other as a subalgebra of the algebra of quasisymmetric functions – and their relation via duality of Hopf algebras. One result is a direct expression for the Kazhdan-Lusztig polynomial of a Bruhat interval in terms of a new invariant, the complete \mathbf{cd} -index. Finally, we summarize the theory of combinatorial Hopf algebras, which gives a unifying framework for the quasisymmetric generating functions developed here.

Mathematics Subject Classification (2010). Primary 06A11; Secondary 05E05, 16T30, 20F55, 52B11.

Keywords. Convex polytope, Eulerian poset, Coxeter group, Kazhdan-Lusztig polynomial, \mathbf{cd} -index, quasisymmetric function, Hopf algebra

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*Partially supported by the US National Science Foundation grant DMS-0555268.

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1. Introduction: Face enumeration in convex polytopes

We begin with an introduction to the enumeration of faces in convex polytopes. For a d -dimensional convex polytope Q , let $f_i = f_i(Q)$ be the number of i -dimensional faces of Q . Thus f_0 is the number of vertices, f_1 the number of edges, \dots , f_{d-1} the number of *facets* (or defining inequalities) of Q . The f -vector of Q is the vector

$$f(Q) = (f_0, f_1, \dots, f_{d-1}).$$

The central problem of this area is to determine when a vector of nonnegative integers $f = (f_0, f_1, \dots, f_{d-1})$ is $f(Q)$ for some d -polytope Q . The case $d = 2$ is clear ($f_0 = f_1 \geq 3$); $d = 3$ was settled by Steinitz in 1906 [54]. The cases $d = 4$ and higher remain open except for special classes of polytopes.

1.1. Simplicial polytopes. A polytope is *simplicial* if all faces are simplices, for example, if its vertices are in general position. Their duals are the *simple* polytopes, which include polytopes with facets in general position. If Q and Q^* are dual d -dimensional polytopes, then their f -vectors are related by $f_i(Q) = f_{d-1-i}(Q^*)$. The f -vectors of simplicial (and, consequently, simple) polytopes have been completely determined.

The h -vector (h_0, \dots, h_d) of a simplicial d -polytope is defined by the polynomial relation

$$\sum_{i=0}^d h_i x^{d-i} = \sum_{i=0}^d f_{i-1} (x-1)^{d-i}. \quad (1)$$

The h -vector and the f -vector of a polytope mutually determine each other via the formulas

$$h_i = \sum_{j=0}^i (-1)^{i-j} \binom{d-j}{i-j} f_{j-1} \quad \text{and} \quad f_{i-1} = \sum_{j=0}^i \binom{d-j}{i-j} h_j,$$

for $0 \leq i \leq d$, so characterizing f -vectors of simplicial polytopes is equivalent to characterizing their h -vectors. This is done in the so-called *g-theorem*, conjectured by McMullen [42] and proved by Billera and Lee [17, 18] (for the sufficiency of the conditions) and Stanley [47] (for their necessity). Given (h_0, \dots, h_d) , define $g_0 := h_0$ and $g_i := h_i - h_{i-1}$ for $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Theorem 1.1. (*g-theorem*): $(h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$ is the h -vector of a simplicial convex d -polytope if and only if

1.1.1. $h_i = h_{d-i}$, for all i ,

1.1.2. $g_0 = 1$, $g_i \geq 0$, for $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$, and

1.1.3. $g_{i+1} \leq g_i^{\langle i \rangle}$ for $i \geq 1$.

The relations in (1.1.1) are known as the *Dehn-Sommerville equations* and date to the early 20th century. The nonnegativity relations (1.1.2) are known as the *generalized lower bound conditions*. These plus the inequalities (1.1.3) are known as the *Macaulay conditions*. The quantity $g_i^{\langle i \rangle}$ is computed by expressing g_i canonically as the sum of a sequence of binomial coefficients and altering them by adding 1 to the top and bottom of each. See [18] for details.

Conditions (1.1.2) and (1.1.3) characterize sequences of natural numbers that count monomials in an order ideal of monomials (a set of monomials closed under the division order). They are similar to, but are not quite the same as, the conditions of Kruskal and Katona for f -vectors of general simplicial complexes, but with g_i in place of f_i . Equivalently, (1.1.2) and (1.1.3) say the g_i 's form the Hilbert function of some graded algebra. The necessity proof of Stanley [47] proceeds by producing a commutative ring with this Hilbert function. See, for example, [11] for complete definitions and references.

1.2. Counting flags in polytopes. For general convex polytopes, the situation for f -vectors is much less satisfactory. In particular, the only equation they all satisfy is the Euler relation

$$f_0 - f_1 + f_2 - \dots \pm f_{d-1} = 1 - (-1)^d.$$

Already in $d = 4$, we do not know all linear inequalities on f -vectors, and at this point, there is little hope of giving an analog to the Macaulay conditions.

A possible solution is to try to solve a harder problem: count not *faces*, but *chains* of faces. For a d -dimensional polytope Q and a set S of possible dimensions, define $f_S(Q)$ to be the number of chains of faces of Q having dimensions prescribed by the set S . The function

$$S \mapsto f_S(Q)$$

is called the flag f -vector of Q . It was first studied by Stanley in the context of balanced simplicial complexes, a natural extension of order complexes of graded posets [46].

The flag f -vector of a polytope includes the ordinary f -vector, by counting chains of one element: $(f_S : |S| = 1)$. It also has a straightforwardly defined flag h -vector that turns out to be a finely graded Hilbert function. Most important for an algebraic approach to flag f -vectors, they satisfy an analog of the Dehn-Sommerville equations, which cut their dimension down to the Fibonacci numbers, compared to $\lfloor \frac{n}{2} \rfloor$ for f -vectors of simplicial polytopes.

In what follows, we discuss the development of the theory of flag vectors of polytopes, and where it has led. We thank Margaret Bayer, Saúl Blanco and Stephanie van Willigenburg for reading and offering corrections on earlier drafts of this paper.

2. Eulerian posets and the cd-index

The best setting in which to study the flag f -vector of a d -polytope Q is that of its lattice of faces $P = \mathcal{F}(Q)$, an Eulerian graded poset of rank $d + 1$. We define the **cd-index** and g -polynomials for Eulerian posets and discuss inequalities on these for polytopes and certain spherical subdivisions.

2.1. Flag enumeration in graded posets. A *graded* poset is a poset P with elements $\widehat{0}$ and $\widehat{1}$ such that $\widehat{0} \leq x \leq \widehat{1}$ for all $x \in P$ and with rank function $\rho : P \rightarrow \mathbb{N}$ so that for each $x \in P$, $\rho(x)$ is the length k of any maximal chain $\widehat{0} = x_0 < x_1 < \cdots < x_k = x$. The *rank* of P is $\rho(P) := \rho(\widehat{1})$.

The *flag f -vector* of a graded poset P of rank $n + 1$ is the function $S \mapsto f_S = f_S(P)$, where for $S = \{i_1, \dots, i_k\} \subseteq [n] := \{1, \dots, n\}$,

$$f_S = \left| \{ y_1 < y_2 < \cdots < y_k \mid y_j \in P, \rho(y_j) = i_j \} \right|.$$

Included is the case $S = \emptyset$, where usually $f_\emptyset = 1$, although later we will let f_\emptyset be unspecified.

To begin to understand flag f -vectors of convex polytopes, it might be helpful first to be able to determine all flag f -vectors of graded posets, or at least determine all linear inequalities satisfied by flag f -vectors of graded posets. The former is an analog of the Kruskal-Katona conditions on f -vectors of simplicial complexes and remains open. The latter are analogs of the Dehn-Sommerville and generalized lower bound relations for graded posets. They are completely determined.

First, it is easy to determine that there are no linear equations that hold for the flag f -vectors of all graded posets [19, Proposition 1.1]. For inequalities, the situation is more interesting. For example, for graded posets of rank 4, it can be shown that the inequality

$$f_{\{1,3\}} - f_{\{1\}} + f_{\{2\}} - f_{\{3\}} \geq 0$$

always holds [15, Example 3].

More generally, a subset of the form $\{i, i+1, \dots, i+k\} \in [n]$ is called an *interval*. For an antichain of intervals $\mathcal{I} \subset 2^{[n]}$, define the *blocking family*

$$b[\mathcal{I}] = \{S \subseteq [n] \mid S \cap I \neq \emptyset, \forall I \in \mathcal{I}\}.$$

Theorem 2.1 ([15]). *A linear form $\sum_{S \subseteq [n]} a_S f_S$ satisfies $\sum_S a_S f_S(P) \geq 0$ for all graded posets P of rank $n + 1$ if and only if for all antichains of intervals $\mathcal{I} \subset 2^{[n]}$,*

$$\sum_{S \in b[\mathcal{I}]} a_S \geq 0.$$

Corollary. *The closed convex cone generated by all flag f -vectors of graded posets is polyhedral and has the (Catalan many) extreme rays $e_{\mathcal{I}} = \sum_{S \in b[\mathcal{I}]} e_S$, where $\{e_S \mid S \subseteq [n]\}$ are the unit vectors in \mathbb{R}^{2^n} .*

Example 1. We consider the case of graded posets of rank 3. The flag f -vector in this case is the vector $f = (f_{\emptyset}, f_{\{1\}}, f_{\{2\}}, f_{\{1,2\}})$, and there are 5 extreme rays corresponding to 5 antichains of intervals.

\mathcal{I}	\emptyset	$\{\{1, 2\}\}$	$\{\{1\}, \{2\}\}$	$\{\{1\}\}$	$\{\{2\}\}$
$e_{\mathcal{I}}$	(1, 1, 1, 1)	(0, 1, 1, 1)	(0, 0, 0, 1)	(0, 1, 0, 1)	(0, 0, 1, 1)

2.2. Eulerian posets and the cd-index. A graded poset P is said to be *Eulerian* if for all $x \leq y \in P$,

$$\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$$

where μ is the Möbius function of P . Equivalently, P is Eulerian if for each subinterval $[x, y] \subseteq P$, the number of elements of even rank is equal to number of elements of odd rank. Face posets of polytopes and spheres are Eulerian.

Again, two natural problems arise, to determine all flag f -vectors of Eulerian posets or, at least, to determine all linear inequalities satisfied by flag f -vectors of Eulerian posets. Here, all the linear equations are known. There are 2^n flag numbers $f_S, S \subseteq [n]$, for graded posets of rank $n + 1$. For Eulerian posets, these are not independent evaluations. In fact, for Eulerian posets, only Fibonacci many f_S are linearly independent.

We consider the first few cases. Note that we consider f_{\emptyset} to be variable, which will be important later for several reasons.

$n = 0$: f_{\emptyset} is the only flag number.

$n = 1$: $f_{\emptyset}, f_{\{1\}}$ are the relevant flag numbers, but $f_{\{1\}} = 2f_{\emptyset}$ (Euler relation).

$n = 2$: $f_{\emptyset}, f_{\{1\}}, f_{\{2\}}, f_{\{1,2\}}$ are all the flag numbers, but $f_{\{1\}} = f_{\{2\}}$ (Euler relation) and $f_{\{1,2\}} = 2f_{\{2\}}$.

$n = 3$: $f_{\emptyset}, f_{\{1\}}, f_{\{2\}}, f_{\{3\}}, f_{\{1,2\}}, f_{\{1,3\}}, f_{\{2,3\}}, f_{\{1,2,3\}}$ are the flag numbers, but $f_{\{1\}} - f_{\{2\}} + f_{\{3\}} = 2f_{\emptyset}$ (Euler relation), $f_{\{1,2\}} = 2f_{\{2\}}$, $f_{\{2,3\}} = 2f_{\{2\}}$, $f_{\{1,3\}} = f_{\{2,3\}}$ and $f_{\{1,2,3\}} = 2f_{\{1,3\}}$. Only $f_{\emptyset}, f_{\{1\}}$ and $f_{\{2\}}$ are independent.

$n = 4$: Only $f_{\emptyset}, f_{\{1\}}, f_{\{2\}}, f_{\{3\}}, f_{\{1,3\}}$ are independent.

Thus the first few dimensions of the linear space spanned by all flag numbers of Eulerian posets of rank $n + 1$ are 1, 1, 2, 3 and 5. The relevant relations for P are all derived from Euler relations in P and in intervals $[x, y]$ of P . Details of these equations will appear later.

There is much less known about inequalities for flag numbers of Eulerian posets. The cones of all flag vectors are known for Eulerian posets through rank 6. The best references for this are [8, 9].

For $S \subseteq [n]$ let the *flag h -vector* be defined by

$$h_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T.$$

For noncommuting indeterminates \mathbf{a} and \mathbf{b} let $u_S = u_1 u_2 \cdots u_n$ be defined by

$$u_i = \begin{cases} \mathbf{b} & \text{if } i \in S \\ \mathbf{a} & \text{if } i \notin S. \end{cases}$$

Let $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. Then for Eulerian posets, the generating function

$$\Psi_P = \sum_S h_S(P) u_S \tag{2}$$

is always a polynomial in \mathbf{c} and \mathbf{d} ; this polynomial $\Phi_P(\mathbf{c}, \mathbf{d})$ is called the *\mathbf{cd} -index* of P . This invariant was first explicitly defined by Bayer and Klapper in [6], following an unpublished suggestion of J. Fine.

Example 2. Let $P = \mathcal{B}_3$, the Boolean algebra of rank 3, *i.e.*, the poset of all subsets of a 3-element set ordered by inclusion. We have $f_\emptyset = 1$, $f_{\{1\}} = 3$, $f_{\{2\}} = 3$, and $f_{\{1,2\}} = 6$ so $h_\emptyset = 1$, $h_{\{1\}} = 2$, $h_{\{2\}} = 2$, $h_{\{1,2\}} = 1$, and

$$\begin{aligned} \Psi_P &= \mathbf{aa} + 2\mathbf{ba} + 2\mathbf{ab} + \mathbf{bb} \\ &= (\mathbf{a} + \mathbf{b})^2 + (\mathbf{ab} + \mathbf{ba}) \\ &= \mathbf{c}^2 + \mathbf{d} = \Phi_P \end{aligned}$$

Another invariant for Eulerian posets that implicitly enumerates flags is the following extension of the h -vector and associated g -vector defined in §1.1. This definition, originally due to MacPherson in the context of convex polytopes and their associated toric varieties, was given in the context of Eulerian posets by Stanley in [48]. For an Eulerian poset P of rank $n + 1 \geq 0$, we define two polynomials $f(P, x), g(P, x) \in \mathbb{Z}[x]$ recursively as follows. If $n + 1 = 0$, then $f(P, x) = g(P, x) = 1$. If $n + 1 > 0$, then

$$f(P, x) = \sum_{y \in P \setminus \{\hat{1}\}} g([\hat{0}, y], x) (x - 1)^{n - \rho(y)}. \tag{3}$$

If $f(P, x) = \sum_{i=0}^n \kappa_i x^i$ has been defined, then we define

$$g(P, x) = \kappa_0 + (\kappa_1 - \kappa_0) x + \cdots + \left(\kappa_{\lfloor \frac{n}{2} \rfloor} - \kappa_{\lfloor \frac{n}{2} \rfloor - 1} \right) x^{\lfloor \frac{n}{2} \rfloor}. \tag{4}$$

For an Eulerian poset P , the vector $(h_0, \dots, h_n) = (\kappa_n, \dots, \kappa_1, \kappa_0)$ is what is sometimes called the *toric h -vector* of P . When P is the face poset of a simplicial polytope (or any simplicial complex), this toric h -vector coincides with the usual simplicial h -vector defined in (1). Since for Eulerian P , $h_i = h_{n-i}$ (see [48] or [51, Theorem 3.14.9]), our definition of $g(P, x)$ agrees with the usual notion of the simplicial g -vector.

That the toric h and g -vectors are functions of the flag f -vector was first noted by Bayer [3]. Formulas expressing these in terms of the flag f -vector (for general graded posets) and the \mathbf{cd} -index (for Eulerian posets) are given in [7]. We note that in [7], this distinction between κ_i and h_i is not made, so their formulas for h_i are, in reality, for h_{n-i} (which equals h_i in the Eulerian case).

2.3. Inequalities for flags in polytopes and spheres. There are by now many inequalities known to hold for the g -polynomial and the \mathbf{cd} -index of convex polytopes and more general spheres. These all give inequalities on the flag f -vectors of these objects. We summarize most of these here.

- Among all n -dimensional polytopes, the g -polynomial is termwise minimized on the n -simplex Δ_n . Since always $g_0 = 1$, this is equivalent to saying that $g_i \geq 0$ for $i \geq 1$ (the generalized lower bound theorem). This was proved by Stanley in [47] and [48] for simplicial and then all rational polytopes using the cohomology of toric varieties, and extended to all polytopes by Karu [37], by means of the theory of combinatorial intersection cohomology. See [21] or [53] for a discussion of this combinatorial cohomology theory.

- For polytopes and, in fact, for all Cohen-Macaulay graded posets (so for face posets of balls and spheres), $h_S \geq 0$ (Stanley, [46]).

- If we write $\Phi_P = \sum_w [w]_P w$ over \mathbf{cd} -words w , then $[w]_P \geq 0$ for polytopes (more generally for S -shellable CW -spheres; Stanley [49]).

- Among all n -dimensional zonotopes, the \mathbf{cd} -index is termwise minimized on the n -cube C_n . Equivalently, among all decompositions of the $(n-1)$ -sphere induced by central hyperplane arrangements in \mathbb{R}^n , the \mathbf{cd} -index is termwise minimized by the n -dimensional crosspolytope (Billera, Ehrenborg and Readdy [14]).

- Among all n -dimensional polytopes, the \mathbf{cd} -index is termwise minimized on the n -simplex Δ_n (Billera and Ehrenborg [13]).

- If Q is a polytope, then termwise as polynomials

$$g(Q) \geq g(F) \cdot g(Q/F)$$

for any any face $F \subseteq Q$, where Q/F is any polytope whose face lattice is the interval $[F, Q]$. This was shown by Braden and MacPherson [22] for rational polytopes using cohomology of toric varieties. Again, it follows for all polytopes by combinatorial intersection cohomology; see [21] for a discussion of this.

- For *any* polytope Q and face $F \subseteq Q$, we have termwise as \mathbf{cd} -polynomials,

$$\Phi_Q \geq \begin{cases} \mathbf{c} \cdot \Phi_F \cdot \Phi_{Q/F} \\ \Phi_F \cdot \mathbf{c} \cdot \Phi_{Q/F} \\ \Phi_F \cdot \Phi_{Q/F} \cdot \mathbf{c}, \end{cases}$$

where $\Phi_Q, \Phi_F, \Phi_{Q/F}$ are the \mathbf{cd} -indices of (the face lattices of) $Q, F, P/F$, respectively (Billera and Ehrenborg [13]).

- For a polytope Q , let $[w]_Q$ denote the coefficient of the **cd**-word w in the **cd**-index of Q . Then for all **cd**-words u and v

$$[u\mathbf{d}v]_Q \geq [u\mathbf{c}^2v]_Q$$

(Ehrenborg [31]).

- If Q is an n -dimensional polytope with v vertices, then for any S ,

(a) $f_S(Q) \leq f_S(C(v, n))$,

(b) $h_S(Q) \leq h_S(C(v, n))$ and

(c) $\Phi_Q \leq \Phi_{C(v, n)}$,

where $C(v, n)$ is the cyclic n -polytope with v vertices, *i.e.*, the convex hull of v points on the moment curve (t, t^2, \dots, t^n) . This is known as the Upper Bound Theorem. The first inequality for the case $|S| = 1$ was proved by McMullen [41] by proving the first two inequalities for all simplicial polytopes in this case. The latter result was extended to all triangulated spheres by Stanley [45]. The first inequality for general S was observed by Bayer and Billera [4]. In full generality, this result is due to Billera and Ehrenborg [13].

- For P a Gorenstein* poset (*i.e.*, one that is both Eulerian and Cohen-Macaulay), $\Phi_P \geq 0$. Gorenstein* posets include all face-posets of regular CW -spheres. This result was conjectured by Stanley in [49] and proved by Karu [38, 39].

- For P a Gorenstein* lattice of rank $n + 1$, Φ_P is bounded below termwise by the **cd**-index of the n -dimensional simplex. This generalizes the result of Billera-Ehrenborg for **cd**-indices of n -dimensional polytopes. This result was also conjectured by Stanley [50] and was proved by Ehrenborg and Karu [32].

There is one outstanding conjecture of Stanley in this area that remains open. What follows is Conjecture 4.2(d) in [48]. The second part is Conjecture 4.3 in [50]. It covers, in particular, g -polynomials of all triangulated spheres. (That the h -polynomial of a triangulated sphere is nonnegative is a consequence of the Cohen-Macaulayness of its face ring [45].)

Conjecture 1 ([48]). For P a Gorenstein* lattice, the g -polynomial, and so the h -polynomial, is nonnegative.

We should note here that there is no guarantee in any of these cases that there are only finitely many irredundant linear inequalities, although in none of these cases have more than finitely many been found. In a related instance, however, Nyman [43] has found that for rank 3 geometric lattices, countably many linear inequalities are necessary to describe their flag f -vectors.

3. Algebraic approaches to counting flags

In this section, we will consider two different algebras that arise in the study of flag f -vectors of graded posets. In the end, we will see that these algebras are, in fact, directly related to each other via duality of Hopf algebras. Especially interesting is how each one handles the case of Eulerian posets.

3.1. The convolution product and derived inequalities. We will write $f_S^{(n)}$, $S \subseteq [n-1]$, when counting chains in a poset of rank n , and we consider $f_S^{(n)}(\cdot)$ to be an operator on posets of rank n . Alternatively, we can define $f_S^{(n)}(P) \equiv 0$ when the rank of P is not n .

Given $f_S^{(n)}$ and $f_T^{(m)}$, $S \subseteq [n-1]$, $T \subseteq [m-1]$ and P a poset of rank $n+m$, define

$$f_S^{(n)} * f_T^{(m)}(P) = \sum_{x \in P : \rho(x)=n} f_S^{(n)}([\widehat{0}, x]) \cdot f_T^{(m)}([x, \widehat{1}]).$$

It is easy to see that $f_S^{(n)} * f_T^{(m)} = f_{S \cup \{n\} \cup (T+n)}^{(n+m)}$, where $T+n := \{x+n \mid x \in T\}$. For example, $f_{\{1\}}^{(2)} * f_{\{2\}}^{(3)} = f_{\{1,2,4\}}^{(5)}$ and $f_{\emptyset}^{(2)} * f_{\emptyset}^{(3)} = f_{\{2\}}^{(5)}$.

This convolution product was first considered by Kalai [36], who used it to produce new flag vector inequalities for polytopes from known ones. It is immediate, that this works as well for graded posets or for Eulerian posets (in fact, for any class of posets closed under taking intervals).

Proposition 3.1 ([36]). *If the linear forms $G_1 = \sum \alpha_S f_S^{(n)}$ and $G_2 = \sum \beta_S f_S^{(m)}$ satisfy $G_1(P_1) \geq 0$ and $G_2(P_2) \geq 0$ for all polytopes (respectively, graded posets, Eulerian posets) P_1 and P_2 of ranks n and m , then $G_1 * G_2(P) \geq 0$ for all polytopes (graded posets, Eulerian posets) P of rank $n+m$.*

Example 3. Polygons have at least 3 vertices, so $f_{\{1\}}^{(3)} - 3f_{\emptyset}^{(3)} \geq 0$ for all polygons. (Note that rank is one more than dimension, so $f_{\{1\}}^{(3)}$ counts *vertices*.) Thus

$$\left(f_{\{1\}}^{(3)} - 3f_{\emptyset}^{(3)} \right) * f_{\emptyset}^{(1)} = f_{\{1,3\}}^{(4)} - 3f_{\{3\}}^{(4)} \geq 0$$

for all 3-polytopes (*i.e.*, the number of vertices in 2-faces is at least three times the number of 2-faces).

Most of the inequalities described earlier are of the form

$$G(P) = \sum \alpha_S f_S^{(n)}(P) \geq 0$$

and so can be convolved to give further inequalities. As an example we consider the coefficients of the **cd**-index. Let $w = \mathbf{c}^{n_1} \mathbf{d} \mathbf{c}^{n_2} \mathbf{d} \mathbf{c}^{n_3} \dots \mathbf{c}^{n_p} \mathbf{d} \mathbf{c}^{n_{p+1}}$ be a **cd**-word, and define m_0, \dots, m_p by $m_0 = 1$ and $m_i = m_{i-1} + n_i + 2$. Then the coefficient of w in the **cd**-index is given by

$$[w] = \sum_{i_1, \dots, i_p} (-1)^{(m_1 - i_1) + (m_2 - i_2) + \dots + (m_p - i_p)} k_{\{i_1 i_2 \dots i_p\}}, \quad (5)$$

where the sum is over all p -tuples (i_1, i_2, \dots, i_p) such that $m_{j-1} \leq i_j \leq m_j - 2$ and

$$k_S = \sum_{T \subseteq S} (-2)^{|S|-|T|} f_T.$$

Using (5), we can see the \mathbf{cd} -indices for Eulerian posets of ranks 1–5 are

$$\begin{aligned} & f_\emptyset^{(1)} \\ & f_\emptyset^{(2)} \mathbf{c} \\ & f_\emptyset^{(3)} \mathbf{c}^2 + (f_{\{1\}}^{(3)} - 2f_\emptyset^{(3)}) \mathbf{d} \\ & f_\emptyset^{(4)} \mathbf{c}^3 + (f_{\{1\}}^{(4)} - 2f_\emptyset^{(4)}) \mathbf{dc} + (f_{\{2\}}^{(4)} - f_{\{1\}}^{(4)}) \mathbf{cd} \\ & f_\emptyset^{(5)} \mathbf{c}^4 + (f_{\{1\}}^{(5)} - 2f_\emptyset^{(5)}) \mathbf{dc}^2 + (f_{\{2\}}^{(5)} - f_{\{1\}}^{(5)}) \mathbf{cdc} + (f_{\{3\}}^{(5)} - f_{\{2\}}^{(5)} + f_{\{1\}}^{(5)} - 2f_\emptyset^{(5)}) \mathbf{c}^2 \mathbf{d} \\ & \quad + (f_{\{1,3\}}^{(5)} - 2f_{\{3\}}^{(5)} - 2f_{\{1\}}^{(5)} + 4f_\emptyset^{(5)}) \mathbf{d}^2, \end{aligned}$$

so, for example, we know from the nonnegativity of the \mathbf{cd} -index that

$$f_{\{1,3\}}^{(5)} - 2f_{\{3\}}^{(5)} - 2f_{\{1\}}^{(5)} + 4f_\emptyset^{(5)} \geq 0$$

for all 4-dimensional convex polytopes.

We remark here that Stenson [56] has shown that the set of inequalities on polytopes derived by convolution from the nonnegativity of the g_i and the set derived from the fact that $[w]$ is bounded below by its value on the simplex do not imply each other.

3.2. Relations on flag numbers and the enumeration algebra.

Eulerian posets of rank d , as well as polytopes of dimension $d - 1$, satisfy the Euler relations

$$f_\emptyset^{(d)} - f_{\{1\}}^{(d)} + f_{\{2\}}^{(d)} - \dots + (-1)^{d-1} f_{\{d-1\}}^{(d)} + (-1)^d f_\emptyset^{(d)} = 0.$$

Since by Proposition 3.1, the convolution product preserves equalities, we can convolve the trivially nonnegative forms $f_S^{(k)}$ with Euler relations to get relations for posets of higher ranks of the form

$$f_S^{(k)} * \left(f_\emptyset^{(d)} - f_{\{1\}}^{(d)} + f_{\{2\}}^{(d)} - \dots + (-1)^{d-1} f_{\{d-1\}}^{(d)} + (-1)^d f_\emptyset^{(d)} \right) * f_T^{(l)} = 0. \quad (6)$$

These are enough to generate all linear relations on flag f -vectors on polytopes.

Theorem 3.2 ([5]). *All linear relations on the $f_S^{(d)}$ for polytopes, and so for Eulerian posets, are derived from those coming from the Euler relations as in (6).*

The equations in [5] are identical to those in equation (6), although they originally were expressed without the use of the convolution. The proof there that these are all the equations consists of producing Fibonacci many polytopes whose flag f -vectors span. These can be made for each dimension by considering repeated operations of forming *pyramids* P and *prisms* B starting with an edge, never taking two B 's in a row. The number of words of length $d - 1$ in P and B , with no repeated B , is a Fibonacci number. A simpler algebraic proof that flag f -vectors of polytopes span that does not give a specific basis is given in [14], where it is shown also that zonotopes will suffice. See also [36] for another basis.

There is a simple algebraic way of capturing the notion of convolution product and relations on flag numbers in Eulerian posets. Let

$$A = \mathbb{Q}\langle y_1, y_2, \dots \rangle = A_0 \oplus A_1 \oplus A_2 \cdots$$

be the free associative \mathbb{Q} -algebra on noncommuting y_i , graded by $\deg(y_i) = i$. Here

$$A_n = \text{span}_{\mathbb{Q}}\{ y_{i_1} y_{i_2} \cdots y_{i_k} \mid i_1 + i_2 + \cdots + i_k = n \}.$$

We say $\beta = (\beta_1, \dots, \beta_k)$ is a *composition* of integer $n > 0$ (written $\beta \models n$) if each $\beta_i > 0$ and $|\beta| := \beta_1 + \cdots + \beta_k = n$. There is a simple bijection between compositions of $n + 1$ and subsets of $[n] := \{1, \dots, n\}$ given by

$$\beta = (\beta_1, \dots, \beta_k) \models n + 1 \mapsto S(\beta) := \{\beta_1, \beta_1 + \beta_2, \dots, \beta_1 + \cdots + \beta_{k-1}\} \subseteq [n]$$

and

$$S = \{i_1, \dots, i_{k-1}\} \subseteq [n] \mapsto \beta(S) := (i_1, i_2 - i_1, i_3 - i_2, \dots, n + 1 - i_{k-1}) \models n + 1.$$

We will freely move between indexing by compositions and the associated subsets in the rest of this paper.

Via the association of y_k and $f_{\emptyset}^{(k)}$ and so of

$$y_{\beta} := y_{\beta_1} \cdots y_{\beta_k}, \quad \beta = (\beta_1, \dots, \beta_k) \models n + 1 \quad \text{and} \quad f_{S(\beta)}^{(n+1)} = f_S^{(n+1)}, \quad S \subseteq [n],$$

multiplication in A can be seen to be the analog of Kalai's convolution of flag f -vectors, in which

$$f_S^{(n)} * f_T^{(m)} = f_{S \cup \{n\} \cup (T+n)}^{(n+m)}.$$

Example 4. With this association $f_{\{1\}}^{(3)} = y_1 y_2$ and so

$$f_{\{1\}}^{(3)} * f_{\{1\}}^{(3)} = y_1 y_2 y_1 y_2 = f_{\{1,3,4\}}^{(6)}.$$

In general, we get an association between elements $G \in A_n$ and expressions of the form $\sum_{S \subseteq [n-1]} \alpha_S f_S^{(n)}$. Multiplying a form G in this algebra by y_i on the right (left) corresponds to summing G evaluated on all faces (or links of faces) of corank (rank) i .

For $k \geq 1$ define in A_k the form

$$\chi_k := \sum_{i+j=k} (-1)^i y_i y_j = \sum_{i=0}^k (-1)^i f_i^{(k)},$$

the k^{th} Euler relation, where we take $y_0 = 1$ and $f_0^{(k)} = f_k^{(k)} = f_\emptyset^{(k)}$. Thus in A_4 ,

$$\chi_4 = y_0 y_4 - y_1 y_3 + y_2 y_2 - y_3 y_1 + y_4 y_0 = 2f_\emptyset^{(4)} - f_{\{1\}}^{(4)} + f_{\{2\}}^{(4)} - f_{\{3\}}^{(4)},$$

the Euler relation for posets of rank 4. By Theorem 3.2, the 2-sided ideal

$$I_{\mathcal{E}} = \langle \chi_k : k \geq 1 \rangle \subset A$$

is the space of all relations on Eulerian posets. We define

$$A_{\mathcal{E}} = A/I_{\mathcal{E}},$$

and think of $A_{\mathcal{E}}$ as the algebra of functionals on Eulerian posets. It turns out that it too is a free associative algebra, the algebra of *odd jumps*.

Theorem 3.3 ([19]). *There is an isomorphism of graded algebras,*

$$A_{\mathcal{E}} \cong \mathbb{Q}\langle y_1, y_3, y_5, \dots \rangle,$$

and so $\dim_{\mathbb{Q}}(A_{\mathcal{E}})_n$ is the n^{th} Fibonacci number.

3.3. Quasisymmetric function of a graded poset. Note that the algebras A and $A_{\mathcal{E}}$ discussed in the last section were noncommutative. We can also associate a pair of commutative algebras to the flag vectors of graded and Eulerian posets.

Let $QSym \subset \mathbb{Q}[[x_1, x_2, \dots]]$ be the algebra of all *quasisymmetric functions*

$$QSym := QSym_0 \oplus QSym_1 \oplus \dots$$

where

$$QSym_n := \text{span}_{\mathbb{Q}}\{M_\beta \mid \beta = (\beta_1, \dots, \beta_k) \models n\}$$

and

$$M_\beta := \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \dots x_{i_k}^{\beta_k}.$$

Here $M_0 = 1$ so $QSym_0 = \mathbb{Q}$; otherwise $k > 0$, each $\beta_i > 0$, and $\beta_1 + \dots + \beta_k = n$. Alternatively, $QSym$ consists of all bounded degree power series in $\mathbb{Q}[[x_1, x_2, \dots]]$ such that for all $\beta \models n$, the coefficient of $x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \dots x_{i_k}^{\beta_k}$ is the same as that of $x_1^{\beta_1} x_2^{\beta_2} \dots x_k^{\beta_k}$ whenever $i_1 < \dots < i_k$.

For example, $(1, 2, 1) \models 4$ and $M_{(1,2,1)} = \sum_{i_1 < i_2 < i_3} x_{i_1}^1 x_{i_2}^2 x_{i_3}^1$. We can index also by subsets. For $S \subseteq [n]$, define

$$M_S = M_S^{(n+1)} := M_{\beta(S)},$$

so, for example, if $S = \{1, 3\} \subseteq [3]$ then $\beta(S) = (1, 2, 1) \models 4$ and

$$M_{\{1,3\}} = M_{\{1,3\}}^{(4)} = M_{(1,2,1)}.$$

This basis $\{M_\beta \mid \beta \models n, n \geq 0\}$ is known as the *monomial* basis for $QSym$.

We note that quasisymmetric functions arise naturally as weight enumerators of P -partitions of labeled posets [34]. In this context, a more natural basis arises as weight enumerators of labeled chains,

$$L_S = \sum_{T \supseteq S} M_T.$$

Here $S \subseteq T \subseteq [n]$ and S is the *descent set* of the labeling. This is known as the *fundamental* basis for $QSym$. See [52, §7.19] for further discussion.

We summarize here the basics of the use of quasisymmetric functions in the theory of flag f -vectors of graded posets and, in particular, Eulerian posets. For a finite graded poset P , with rank function $\rho(\cdot)$, we define the formal power series

$$F(P) := \sum_{\hat{0}=u_0 \leq \dots \leq u_{k-1} < u_k = \hat{1}} x_1^{\rho(u_0, u_1)} x_2^{\rho(u_1, u_2)} \dots x_k^{\rho(u_{k-1}, u_k)}, \quad (7)$$

where the sum is over all finite *multichains* in P whose last two elements are distinct and $\rho(x, y) = \rho(y) - \rho(x)$. See [30] for general properties of $F(P)$. In particular, we have the following.

Proposition 3.4. *For a graded poset P ,*

3.4.1. $F(P) \in QSym$, in fact $F(P) \in QSym_n$ where $n = \rho(P)$,

3.4.2. $F(P_1 \times P_2) = F(P_1) F(P_2)$,

3.4.3. $F(P) = \sum_{\alpha} f_{\alpha} M_{\alpha} = \sum_{\alpha} h_{\alpha} L_{\alpha}$, where f_{α} and h_{α} are the flag f and flag h -vectors, respectively, of P .

We define next an interesting subalgebra of $QSym$ that turns out to be related to Eulerian posets. For a **cd**-word w of degree n ,

$$w = \mathbf{c}^{n_1} \mathbf{d} \mathbf{c}^{n_2} \mathbf{d} \dots \mathbf{c}^{n_k} \mathbf{d} \mathbf{c}^m,$$

where $\deg \mathbf{c} = 1$ and $\deg \mathbf{d} = 2$, let

$$\mathcal{I}_w = \{\{i_1 - 1, i_1\}, \{i_2 - 1, i_2\}, \dots, \{i_k - 1, i_k\}\},$$

where $i_j = \deg(\mathbf{c}^{n_1} \mathbf{d} \mathbf{c}^{n_2} \mathbf{d} \dots \mathbf{c}^{n_j} \mathbf{d})$. Note that \mathcal{I}_w consists of disjoint intervals in $[n]$, all of size 2. These and more general *even* antichains of intervals have been related to extremes of the cone of Eulerian flag vectors in [8, 9].

The *peak algebra* Π is defined to be the subalgebra of $QSym$ generated by the *peak quasisymmetric functions*

$$\Theta_w = \sum_{T \in b[\mathcal{I}_w]} 2^{|T|+1} M_T^{(n+1)}, \quad (8)$$

where w is any **cd**-word (including empty **cd**-word $\mathbf{1}$, for which $\mathcal{I}_1 = \emptyset$). Note that if $\deg w = n$, then $\deg \Theta_w = n + 1$; there are Fibonacci many Θ_w of each degree.

The peak algebra was first defined by Stembridge [55], where peak quasisymmetric functions arise naturally as weight enumerators of *enriched P-partitions* of labeled posets.

A subset $S \subseteq [n]$ is *sparse* if $1 \notin S$ and $i \in S \Rightarrow i - 1 \notin S$. We can associate a sparse subset $S_w \subseteq [n]$ to a **cd**-word of degree n by associating

$$w = \mathbf{c}^{n_1} \mathbf{d} \mathbf{c}^{n_2} \mathbf{d} \cdots \mathbf{c}^{n_k} \mathbf{d} \mathbf{c}^m \quad \text{and} \quad S_w = \{i_1, i_2, \dots, i_k\} \subset [n],$$

where $i_j = \deg(\mathbf{c}^{n_1} \mathbf{d} \mathbf{c}^{n_2} \mathbf{d} \cdots \mathbf{c}^{n_j} \mathbf{d})$. Stembridge considers the basis for Π to be labeled by sets S of the form S_w . In this context, his basis Θ_S arises as weight enumerators of labeled chains, where S is the *peak set* of the labeling. (A *peak* is a descent preceded by an ascent.)

3.4. Peak functions and Eulerian posets. The main result for our purposes with respect to the subalgebra Π is due to Bergeron, Mykytiuk, Sottile and van Willigenburg [10].

Theorem 3.5. *If P is an Eulerian poset, then $F(P) \in \Pi$.*

The proof of Theorem 3.5 depends on connections between the enumeration algebra $\mathbb{Q}\langle y_1, y_2, \dots \rangle$ and the algebra of quasisymmetric functions $QSym$ as well as between the quotient $A_{\mathcal{E}}$ and the subalgebra Π of peak functions. Now the algebras Π and $A_{\mathcal{E}}$ both have Hilbert series given by the Fibonacci sequence, although they are surely not isomorphic: Π is commutative, $A_{\mathcal{E}}$ is not. The connection comes via duality of Hopf algebras. We summarize this briefly here.

Let B be a graded algebra. The product on the algebra B can be viewed as a homogeneous linear map

$$B \otimes B \longrightarrow B, \quad a \otimes b \mapsto a \cdot b$$

A coalgebra C has instead a *coproduct* $C \longrightarrow C \otimes C$, as well as a *counit*, an analog of the unit in an algebra. A *Hopf algebra* H has both product and coproduct (plus unit and counit), as well as a map $S : H \rightarrow H$ known as the *antipode*. (In the case of graded Hopf algebras, the antipode is uniquely specified by the product and coproduct; see, *e.g.* [30, Lemma 2.1] or the Appendix in [10].) In the dual vector space H^* to a Hopf algebra H , the adjoint of the product on H

$$H^* \otimes H^* \longleftarrow H^*$$

gives a coproduct on H^* , and the adjoint of the coproduct on H

$$H^* \longleftarrow H^* \otimes H^*$$

gives a product on H^* , making H^* a Hopf algebra as well. H^* is the *dual Hopf algebra* to H .¹

¹In reality, we are considering the *graded dual* $H^* = \oplus H_i^*$ of the graded Hopf algebra $H = \oplus H_i$ [2]. All products and coproducts we describe will be homogeneous maps.

The four algebras we have discussed are all graded Hopf algebras, with the coproducts defined below. In [33], the *integral* Hopf algebra $NC = \mathbb{Z}\langle y_1, y_2, \dots \rangle$ (called there the *noncommutative symmetric functions*) was shown to be dual to the Hopf algebra of quasisymmetric functions with integral coefficients, with coproducts

$$\Delta(M_\beta) = \sum_{\beta = \beta_1 \cdot \beta_2} M_{\beta_1} \otimes M_{\beta_2}$$

for $QSym$ and

$$\Delta(y_k) = \sum_{i+j=k} y_i \otimes y_j.$$

for NC . So, for example,

$$\Delta(M_{(2,1,1)}) = 1 \otimes M_{(2,1,1)} + M_{(2)} \otimes M_{(1,1)} + M_{(2,1)} \otimes M_{(1)} + M_{(2,1,1)} \otimes 1$$

and $\Delta(y_2) = 1 \otimes y_2 + y_1 \otimes y_1 + y_2 \otimes 1$, where, as before, we take $y_0 = 1$.

In [10], these coproducts on $QSym$ and A , respectively, are shown to extend to coproducts on Π and $A_{\mathcal{E}}$, and they proved [10, Theorem 5.4]:

Theorem 3.6 ([10]). *These coproducts make Π and $A_{\mathcal{E}}$ into a dual pair of Hopf algebras.*

Theorem 3.5 follows directly from this: For any graded poset P , the quasisymmetric function $F(P) = \sum_S f_S(P) M_S$ defines a functional $A \rightarrow \mathbb{Q}$, defined by $\sum_S \alpha_S f_S \mapsto \sum_S \alpha_S f_S(P)$, in $A^* = QSym$. Theorem 3.6 implies that Π is the kernel of the restriction of this functional to functionals on the ideal $I_{\mathcal{E}}$. By the definition of $I_{\mathcal{E}}$, any Eulerian P has an $F(P)$ in this kernel, so $F(P) \in \Pi$.

This leads immediately to the following question: For an Eulerian poset P , what is the representation of $F(P)$ in terms of the basis of peak functions $\{\Theta_w\}$ for Π ? Equivalently, what is the dual basis in $A_{\mathcal{E}}$ to the basis $\{\Theta_w\}$? This was answered in [16].

Theorem 3.7 ([16]). *If P is any Eulerian poset, then*

$$F(P) = \sum_w \frac{1}{2^{|w|_{\mathbf{d}}+1}} [w]_P \Theta_w,$$

where the $[w]_P$ are the coefficients of the **cd**-index of P and $|w|_{\mathbf{d}}$ is the number of **d**'s in w .

Corollary. *The elements*

$$\frac{1}{2^{|w|_{\mathbf{d}}+1}} [w] \in A_{\mathcal{E}}$$

form a dual basis to the basis Θ_w in Π .

Since, in terms of the theory of P -partitions, the subalgebra Π and the basis $\{\Theta_w\}$ arise naturally when considering the algebra $QSym$, one sees that the **cd**-index is a natural, in fact, inescapable, invariant in the context of flag enumeration in Eulerian posets. We see in the next section how these ideas lead to an interesting new invariant in the theory of Bruhat intervals on Coxeter groups.

4. Bruhat intervals in Coxeter groups

A Coxeter group is a group W generated by a finite set S with the relations $s^2 = e$ for all $s \in S$ (e is the identity of W) and otherwise only relations of the form

$$(ss')^{m(s,s')} = e,$$

for $s \neq s' \in S$ with $m(s, s') = m(s', s) \geq 2$. There are many examples of such groups, including the symmetry groups of regular polytopes (and so the symmetric groups) and the finite reflection groups. See [35] and [20] for general background, especially the latter for the combinatorial theory of Coxeter groups discussed here.

Given a Coxeter system (W, S) (the set of generators is a critical component), each $v \in W$ can be written $v = s_1 s_2 \cdots s_k$ with $s_i \in S$. If k is minimal among all such expressions for v , then $s_1 s_2 \cdots s_k$ is called a *reduced expression* for v and $k = l(v)$ is called the *length* of v .

The Bruhat order on (W, S) is a partial order on the set W , defined as follows. If $v = s_1 s_2 \cdots s_k$ is a reduced expression for v , then $u \leq v$ for $u \in W$ if some (reduced) expression for u is a subword $u = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, $i_1 < i_2 < \cdots < i_\ell$, of v .

It was shown by Verma [57] that for each $u \leq v \in W$ the *Bruhat interval* $[u, v]$ is an Eulerian poset of rank $l(u, v) := l(v) - l(u)$. Thus, as an Eulerian poset, the interval $[u, v]$ has a **cd**-index. This was first studied in any detail by Reading [44], who showed that there were no equations other than those described in Theorem 3.2 that held for the flag vectors of all Bruhat intervals.

Here we extend the **cd**-index of a Bruhat interval to the *complete cd*-index, a nonhomogeneous **cd**-polynomial of degree $l(u, v) - 1$ that includes enough information to compute important invariants for the interval, including its R -polynomial and its Kazhdan-Lusztig polynomial. The remainder of this section represents mostly joint work with Francesco Brenti [12].

4.1. R -polynomial and Kazhdan-Lusztig polynomial. Let $\mathcal{H}(W)$ be the *Hecke algebra* associated to W , i.e. the free $\mathbb{Z}[q, q^{-1}]$ -module having the set $\{T_v \mid v \in W\}$ as a basis and multiplication such that for all $v \in W$ and $s \in S$

$$T_v T_s = \begin{cases} T_{vs}, & \text{if } l(vs) > l(v) \\ qT_{vs} + (q-1)T_v, & \text{if } l(vs) < l(v). \end{cases}$$

Note that were we to set $q = 1$, then this would give precisely the integral group ring of W . $\mathcal{H}(W)$ is an associative algebra having T_e as unity, in which each T_v is invertible. For $v \in W$,

$$(T_{v^{-1}})^{-1} = q^{-l(v)} \sum_{u \leq v} (-1)^{l(u,v)} R_{u,v}(q) T_u,$$

where $R_{u,v}(q) \in \mathbb{Z}[q]$.

The polynomials $R_{u,v}$ are called the *R -polynomials* of W . For $u, v \in W$, $u \leq v$, $\deg(R_{u,v}) = l(u, v)$ and $R_{u,u}(q) = 1$. It is customary to set $R_{u,v}(q) \equiv 0$ if $u \not\leq v$.

The *Kazhdan-Lusztig polynomial* $P_{u,v}$ of a Bruhat interval $[u, v]$ is defined by the following theorem. A proof can be found in [35, §9-11] of an equivalent statement. The version here is [20, Theorem 5.1.4].

Theorem 4.1. *There is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subset \mathbb{Z}[q]$, such that, for all $u, v \in W$,*

4.1.1. $P_{u,v}(q) = 0$ if $u \not\leq v$;

4.1.2. $P_{u,u}(q) = 1$;

4.1.3. $\deg(P_{u,v}(q)) \leq \lfloor \frac{l(u,v)-1}{2} \rfloor$, if $u < v$, and

4.1.4.

$$q^{l(u,v)} P_{u,v} \left(\frac{1}{q} \right) = \sum_{u \leq z \leq v} R_{u,z}(q) P_{z,v}(q),$$

if $u \leq v$.

The main conjectures in this area are that for all Coxeter systems (W, S) and all Bruhat intervals $[u, v]$ in W , the Kazhdan-Lusztig polynomial $P_{u,v}$ is nonnegative, and depends only on the poset $[u, v]$, and not on the underlying group. The first conjecture is known to hold, for example, for all finite Coxeter groups and the second for all *lower* intervals, that is, intervals where $u = e$, the identity element of W [24]. Both conjectures are known to hold when the interval $[u, v]$ is a lattice². See the discussion pp. 161–162 and 171–172 of [20] for references.

4.2. The complete quasisymmetric function of a Bruhat interval and the complete cd-index. While the R -polynomial of a Bruhat interval may have negative terms, there is an associated polynomial that has nonnegative coefficients with a direct combinatorial interpretation. The following is [20, Proposition 5.3.1].

Proposition 4.2. *For $u \leq v \in W$, there exists a (necessarily unique) polynomial $\tilde{R}_{u,v}(q) \in \mathbb{N}[q]$ such that*

$$R_{u,v}(q) = q^{\frac{l(u,v)}{2}} \tilde{R}_{u,v} \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}} \right).$$

For a Bruhat interval $[u, v]$, we use the \tilde{R} -polynomials to define a nonhomogeneous analog of the quasisymmetric function $F(P)$ of a graded poset. For Bruhat interval $[u, v]$, the *complete quasisymmetric function* is defined by

$$\tilde{F}(u, v) := \sum_{u=u_0 \leq \dots \leq u_{k-1} < u_k=v} \tilde{R}_{u_0, u_1}(x_1) \tilde{R}_{u_1, u_2}(x_2) \cdots \tilde{R}_{u_{k-1}, u_k}(x_k). \quad (9)$$

Again, the sum is over all finite multichains in $[u, v]$ whose last two elements are distinct. It is straightforward to show that \tilde{F} is multiplicative [12, Proposition 2.6], that is, for Bruhat intervals $[u_i, v_i]$, $\tilde{F}([u_1, v_1] \times [u_2, v_2]) = \tilde{F}(u_1, v_1) \tilde{F}(u_2, v_2)$.³

²This follows since in this case $P_{u,v}(q) = g([u, v]^*, q)$, which depends only on the poset $[u, v]$ (see Remark 1 in §4.2 and Remark 2 in §4.3). By an unpublished result of Dyer, lattice Bruhat intervals are face posets of polytopes, so nonnegativity follows from the generalized lower bound theorem for polytopes.

³In fact both F and \tilde{F} are maps of Hopf algebras (see [30, Proposition 4.4] and [12, Remark 2.8]). This will also be a consequence of the results discussed in §5.

To give an analog of Proposition 3.4 for $\tilde{F}(u, v)$, we need to define the Bruhat graph of the interval $[u, v]$. Let $T = \{wsw^{-1} \mid w \in W, s \in S\}$ be the set of all conjugates of the generators in W . Elements of T are called *reflections*, while elements of S are called *simple reflections*.

We define the *Bruhat graph* of a Coxeter system (W, S) to be the directed graph $B(W, S)$ obtained by taking W as vertex set and putting a directed edge from x to y if and only if $x^{-1}y \in T$ and $l(x) < l(y)$. We can consider the edge (x, y) of $B(W, S)$ to be labeled by the reflection $t = x^{-1}y$.

The Bruhat graph of an interval $[u, v]$ is the subgraph of $B(W, S)$ induced by the elements in $[u, v]$; it contains the Hasse diagram of the poset (directed in increasing Bruhat order) as a spanning subgraph. The Bruhat graph was first defined by Dyer [28], who showed the graph (not including the labeling) to depend only on the isomorphism class of the poset $[u, v]$ and not on the underlying group.

A *reflection subgroup* of W is any subgroup W' of W generated by a subset of T . For $w \in W$, define $N(w) := \{t \in T : l(tw) < l(w)\}$. Reflection subgroups W' are Coxeter groups, with simple reflections $S' = \{t' \in T : N(t') \cap W' = \{t'\}\}$ [26, 27]. See also [35, §8.2]. A reflection subgroup (W', S') is said to be *dihedral* if $|S'| = 2$.

A total ordering $<_T$ on the set of all reflections T in (W, S) is called a *reflection ordering* if it satisfies the following: For any dihedral reflection subgroup (W', S') , where $S' = \{a, b\}$, either $a <_T aba <_T ababa <_T \cdots <_T babab <_T bab <_T b$ or $b <_T bab <_T babab <_T \cdots <_T ababa <_T aba <_T a$. The existence of reflection orderings for any Coxeter system was shown by Dyer in [29].

Example 5. The symmetric group $W = \mathcal{S}_n$ is a Coxeter group (often denoted A_{n-1}) with Coxeter generators given by the *adjacent* transpositions $s_i = (i \ i+1)$, $i = 1, \dots, n-1$. Here, reflections are *all* transpositions $(i \ j)$, and lexicographic order is a reflection order. Thus in \mathcal{S}_4 , $(12) <_T (13) <_T (14) <_T (23) <_T (24) <_T (34)$.

Given a reflection ordering on the interval $[u, v]$, directed u - v paths in its Bruhat graph are labeled by reflections, and so they have a well-defined descent set in this ordering. For $\alpha \models k$, $k \leq n+1 = l(u, v)$, we denote by $b_\alpha = b_\alpha(u, v)$ the number of paths of length k having descent set $S = S(\alpha)$. Further, define

$$c_\alpha(u, v) = \sum_{\{\beta \models n \mid \alpha \preceq \beta\}} b_\beta(u, v)$$

where \preceq denotes refinement of compositions (parts of β are sums of successive parts of α). Using the quantities b_α and c_α , we can express the complete quasisymmetric function $\tilde{F}(u, v)$ in terms of the fundamental and monomial bases for $QSym$.

Proposition 4.3 ([12]). $\tilde{F}(u, v) = \sum_\alpha c_\alpha(u, v) M_\alpha = \sum_\alpha b_\alpha(u, v) L_\alpha$

Thus we see that $c_\alpha(u, v)$ and $b_\alpha(u, v)$ are analogs of the flag f - and flag h -numbers. Note that it is possible that the quantities $c_\alpha(u, v)$ can be *greater than 1* for $\alpha \models k$, $k < l(u, v)$, that is, there can be more than one rising Bruhat path of less than maximum length.

Since the Bruhat order on $[u, v]$ is always Eulerian, we know $F([u, v]) \in \Pi$, but usually $\tilde{F}(u, v) \neq F([u, v])$. In [23, Theorem 8.4], Brenti showed that the coefficients $c_\alpha(u, v)$ satisfy the equations (6), and so by [16, Proposition 1.3], we can conclude

Theorem 4.4. *For any Bruhat interval $[u, v]$, $\tilde{F}(u, v) \in \Pi$, in fact*

$$\tilde{F}(u, v) \in \Pi_{l(u,v)} \oplus \Pi_{l(u,v)-2} \oplus \Pi_{l(u,v)-4} \oplus \cdots .$$

The last assertion follows since the $b_\alpha(u, v)$ count directed paths from u to v of length $|\alpha|$ in the Bruhat graph $B(W, S)$, and all of these must have length $k \equiv l(u, v) \pmod{2}$. This is true since for any reflection t , $l(wt) - l(w)$ is odd, and so the length of every Bruhat path has the same parity.

Since $\tilde{F}(u, v) \in \Pi$, we can express it in terms of the peak basis Θ_w . We define the *complete cd-index* of the Bruhat interval $[u, v]$

$$\tilde{\Phi}_{u,v} := \sum_w [w]_{u,v} w$$

by the unique expression

$$\tilde{F}(u, v) = \sum_w [w]_{u,v} \left[\frac{1}{2^{|w|_{\mathfrak{d}+1}} \Theta_w \right],$$

where the sum is over all **cd**-words w with $\deg(w) = l(u, v) - 1, l(u, v) - 3, \dots$

In [29], Dyer shows that the polynomial $\tilde{R}_{u,v}(q)$ enumerates *rising* paths in the Bruhat graph of $[u, v]$, *i.e.*, the coefficient of q^k is the number of paths of length k with empty descent set (see [29, Corollary 3.4] or [20, Theorem 5.3.4]). In [29, §4], he also shows that the reflection labeling of the Bruhat graph gives an *EL*-labeling on the maximal length Bruhat paths in $[u, v]$. Together, they imply that the leading term of $\tilde{R}_{u,v}(q)$ is 1, since, in particular, an *EL*-labeling will always have a unique rising path.

Remark 1. One consequence of this is that $c_\alpha(u, v) = f_\alpha([u, v])$ when $\alpha \models l(u, v)$ and so $\tilde{F}(u, v) = F([u, v]) +$ lower terms. Thus the top-degree terms of $\tilde{\Phi}_{u,v}$ (*i.e.*, those of degree $l(u, v) - 1$) constitute the ordinary **cd**-index of the underlying poset $[u, v]$, *i.e.*, $\tilde{\Phi}_{u,v} = \Phi_{[u,v]} +$ lower terms. If $[u, v]$ is a lattice, then $\tilde{\Phi}_{u,v} = \Phi_{[u,v]}$.⁴

By Dyer's *EL*-labeling (or by the earlier *CL*-labeling of Björner and Wachs; see [20, Corollary 2.7.6]), the poset $[u, v]$ is Gorenstein*, so by the result of Karu, $\Phi_{[u,v]} \geq 0$. The following is Conjecture 6.1 in [12].

Conjecture 2 ([12]). For all Bruhat intervals $[u, v]$, $\tilde{\Phi}_{u,v} \geq 0$.

⁴It is a consequence of an unpublished result of Dyer that $\tilde{\Phi}_{u,v} = \Phi_{[u,v]}$ if and only if $[u, v]$ is a lattice.

We can easily see that all the pure \mathbf{c} coefficients $[\mathbf{c}^{k-1}]_{u,v} = b_{(k)}(u,v) \geq 0$, where (k) is the composition with one part. Since $b_{(k)}$ counts the rising paths of length k , we get the following [12, Corollary 2.10].

Proposition 4.5. *For $u < v$, $\tilde{R}_{u,v}(q) = q \tilde{\Phi}_{u,v}(q, 0)$.*

There is some evidence for Conjecture 2; see [12, §6], where one consequence is proved. Further, if d_{min} is the *least* degree of a term in $\tilde{\Phi}_{u,v}$, it is known that if $d_{min} \leq 2$ or if $[\mathbf{c}^{d_{min}}] = 1$, then $[w]_{u,v} \geq 0$ if $\deg w = d_{min}$.⁵

In [44], Reading also showed that for *lower* intervals $[e, v]$, $\Phi_{[e,v]}$ is termwise *less than or equal* to the \mathbf{cd} index of the Boolean algebra $\mathcal{B}_{l(v)}$ of rank $l(v)$. We conjecture that this also bounds the complete \mathbf{cd} -index of lower intervals, in the following sense.

Conjecture 3. For all lower Bruhat intervals $[e, v]$, $\tilde{\Phi}_{e,v}(1, 1) \leq \Phi_{\mathcal{B}_{l(v)}}(1, 1)$.

4.3. Kazhdan-Lusztig polynomial and the complete \mathbf{cd} -index.

We note here that if we were only interested in the complete \mathbf{cd} -index of the interval $[u, v]$, it could have been defined directly by means of a nonhomogeneous \mathbf{ab} polynomial $\tilde{\Psi}_{u,v}$ defined analogously to (2), using the quantities b_α in place of h_S (see [12, Proposition 2.9]). However, the form of the quasisymmetric function $\tilde{\Phi}_{u,v}$ given in Proposition 4.3 leads directly to a way of expressing the Kazhdan-Lusztig polynomial $P_{u,v}$ in terms of the coefficients of the complete \mathbf{cd} -index.

We first consider a family of polynomials $B_k(q)$. We call these *ballot polynomials*, since the coefficient of q^i in $B_k(q)$ is the number of ways k ballots can be cast so that the losing candidate receives i votes, while the winning candidate is never behind. Define

$$B_k(q) := \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k+1-2i}{k+1} \binom{k+1}{i} q^i. \quad (10)$$

The constant term of $B_k(q)$ is always 1 and, when k is even, the lead term is a Catalan number.

For $n \geq 0$ define the *dihedral poset* D_n of rank $n+1$ to be a graded poset with two elements at each rank $1 \leq i \leq n$ where $x \leq y$ if $\rho(x) \leq \rho(y)$. Since each interval in a dihedral poset is dihedral, it is easy to see that D_n is Eulerian for each $n \geq 0$, and it is an easy calculation to see that $\Phi_{D_n} = \mathbf{c}^n$. D_n is the underlying Bruhat poset of a dihedral group of order $2n+2$, and it follows from discussion following [20, Proposition 5.1.8] that $P_{D_n} = 1$. It is straightforward to verify that $\tilde{\Phi}_{D_n} = \mathbf{c} \cdot \tilde{\Phi}_{D_{n-1}} + \tilde{\Phi}_{D_{n-2}}$, with $\tilde{\Phi}_{D_0} = 1$ and $\tilde{\Phi}_{D_1} = \mathbf{c}$, and so $\tilde{\Phi}_{D_n} = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j}{n-2j} \mathbf{c}^{n-2j}$. In fact $\tilde{\Phi}_{u,v}(\mathbf{c}, \mathbf{d}) = \tilde{\Phi}_{u,v}(\mathbf{c}, 0)$ if and only if $[u, v]$ is dihedral.⁵ As for the g -polynomial of D_n , the following is [48, Proposition 2.5].

Proposition 4.6. *The g -polynomial of the dihedral poset D_n is the alternating ballot polynomial $B_n(-q)$.*

⁵These are results that will appear in the forthcoming Cornell Ph.D. Thesis of S.A. Blanco.

In [20, Theorem 5.5.7], an expression is given for $P_{u,v}$, $u < v$, in terms of the $b_\alpha(u, v)$ and universal polynomials Υ_α that enumerate an implicitly defined set of lattice paths. By expressing this in terms of the complete \mathbf{cd} -index of $[u, v]$, the resulting paths are now explicit, and we can get an expression for $P_{u,v}$ in terms of only the coefficients $[w]_{u,v}$ of the complete \mathbf{cd} index $\tilde{\Phi}_{u,v}$ and shifts of the alternating ballot polynomials $B_k(-q)$.

A \mathbf{cd} -word w is said to be *even* if it is a word in \mathbf{c}^2 and \mathbf{d} . For an even \mathbf{cd} -word $w = \mathbf{c}^{n_1} \mathbf{d} \mathbf{c}^{n_2} \mathbf{d} \cdots \mathbf{d} \mathbf{c}^{n_k}$, let $C_w = C_{n_1/2} \cdots C_{n_k/2}$, where $C_i = \frac{1}{2i+1} \binom{2i+1}{i}$, the i^{th} Catalan number. Finally, let $|w| := \deg w$ and $|w|_{\mathbf{d}}$ be the number of \mathbf{d} 's in w . The following is [12, Theorem 4.1].

Theorem 4.7. *For any Bruhat interval $[u, v]$ of rank $l(u, v) = n + 1$,*

$$P_{u,v}(q) = \sum_{i=0}^{\lfloor n/2 \rfloor} a_i q^i B_{n-2i}(-q)$$

where

$$a_i = a_i(u, v) = [\mathbf{c}^{n-2i}]_{u,v} + \sum_{\mathbf{d}w \text{ even}} (-1)^{\frac{|w|}{2} + |w|_{\mathbf{d}}} C_{\mathbf{d}w} [\mathbf{c}^{n-2i} \mathbf{d}w]_{u,v}. \quad (11)$$

Note that the coefficient $a_i(u, v)$ of $q^i B_{n-2i}(-q)$ in this expression for $P_{u,v}$ depends only on \mathbf{cd} -words beginning with \mathbf{c}^{n-2i} that are otherwise even. The expression for $P_{u,v}(q) = p_0 + p_1 q + \cdots$ in terms of the $a_i(u, v)$ can be inverted to give

$$a_j = \sum_{i=0}^j \binom{n-j-i}{n-2j} p_i, \quad (12)$$

for $j = 0, \dots, \lfloor n/2 \rfloor$. Thus if $P_{u,v}(q) \geq 0$ then $a_i(u, v) \geq 0$ for $i = 0, \dots, \lfloor n/2 \rfloor$. The conjectured nonnegativity of $P_{u,v}$ leads to the following, which is [25, Conjecture 6.6] as well as [12, Conjecture 4.11].

Conjecture 4. For each Bruhat interval $[u, v]$ of rank $l(u, v) = n + 1$, $a_i(u, v) \geq 0$ for $i = 0, 1, \dots, \lfloor n/2 \rfloor$.

Remark 2. We note that if we restrict the $[w]_{u,v}$ in (11) to those of degree n only, then we get the formula of Bayer and Ehrenborg [7, Theorem 4.2] for the g -polynomial of the dual poset $[u, v]^*$. Thus the difference $P_{u,v}(q) - g([u, v]^*, q)$ is a function of the lower-degree \mathbf{cd} -coefficients only (and their only function in this expression). Example 4.6 of [12] gives a pair of rank 6 Bruhat intervals in $W = \mathcal{S}_5$ having the same \mathbf{cd} -index but unequal Kazhdan-Lusztig polynomials, and thus unequal complete \mathbf{cd} -indices.

Finally, we point out that as far as combinatorial invariance is concerned, $P_{u,v}$, $a_i(u, v)$ and $[w]_{u,v}$ are all equivalent. We say that an invariant of Bruhat intervals is *combinatorially invariant* if its value on a Bruhat interval $[u, v]$ depends only on the isomorphism type of the poset $[u, v]$.

Proposition 4.8. *The following are equivalent for all Coxeter systems (W, S) .*

4.8.1 *For all $u \leq v \in W$, $P_{u,v}$ is combinatorially invariant.*

4.8.2 *For all $u \leq v \in W$ and $i = 0, \dots, \lfloor \frac{l(u,v)-1}{2} \rfloor$, $a_i(u,v)$ is combinatorially invariant.*

4.8.3 *For all $u \leq v \in W$, and all **cd**-words of degree $n, n-2, \dots$, where $n = l(u,v) - 1$, $[w]_{u,v}$ is combinatorially invariant.*

The equivalence of 4.8.1 and 4.8.3 is discussed in [12, Remark 4.13].

5. Epilog: Combinatorial Hopf Algebras

There is a general enumeration theory that explains the existence of the quasisymmetric functions such as $F(P)$ and $\hat{F}(u,v)$ as well as many other quasisymmetric generating functions that arise in combinatorial theory. Originally formulated by Aguiar in [1] in the context of infinitesimal Hopf algebras, it was later expanded by Aguiar, Bergeron and Sottile and reformulated for Hopf algebras [2]. We summarize this theory and a more recent extension below.

Let $H = H_0 \oplus H_1 \oplus H_2 \oplus \dots$ be a graded connected Hopf algebra (say, over \mathbb{Q}). This means $H_0 \cong \mathbb{Q}$ and the product and coproduct are homogeneous maps. A *character* of H is an algebra morphism $\zeta : H \rightarrow \mathbb{Q}$, and the pair (H, ζ) is called a *combinatorial Hopf algebra*. A morphism $f : (H', \zeta') \rightarrow (H, \zeta)$ of combinatorial Hopf algebras is a morphism of graded Hopf algebras $f : H' \rightarrow H$ such that $\zeta' = \zeta \circ f$.

Example 6. Let \mathcal{P} be the \mathbb{Q} -vector space with basis consisting of all isomorphism classes of graded posets. We define a product on \mathcal{P} by $P_1 \cdot P_2 := P_1 \times P_2$, the Cartesian product of posets, and coproduct by $\Delta(P) = \sum_{x \in P} [\hat{0}, x] \otimes [x, \hat{1}]$. The unit element of \mathcal{P} is the poset $\mathbf{1}$ with one element $\hat{0} = \hat{1}$, and the counit is $\epsilon(P) = \delta_{P, \mathbf{1}}$. See, for example, [30]. If we take ζ to be the usual zeta function for posets, defined by $\zeta(P) = 1$ for all posets P , the pair (\mathcal{P}, ζ) is called the *combinatorial Hopf algebra of posets* [2].

The Hopf algebra $QSym$ becomes a combinatorial Hopf algebra with the canonical character $\zeta_{\mathbb{Q}}$ defined by $\zeta_{\mathbb{Q}}(M_{\alpha}) = 1$ if $\alpha = (n)$, $n \geq 0$, $\zeta_{\mathbb{Q}}(M_{\alpha}) = 0$ otherwise. The main result [2, Theorem 4.1] is that the combinatorial Hopf algebra $(QSym, \zeta_{\mathbb{Q}})$ is a terminal object in the category of combinatorial Hopf algebras, that is, for any combinatorial Hopf algebra (H, ζ_H) , there is a unique morphism $F : (H, \zeta_H) \rightarrow (QSym, \zeta_{\mathbb{Q}})$. For $(H, \zeta_H) = (\mathcal{P}, \zeta)$ from Example 6, the morphism F is the one given in (7).

Further, each combinatorial Hopf algebra (H, ζ_H) has a special subalgebra Π_H , called the *odd subalgebra*, and the morphism F satisfies $F(\Pi_H) \subseteq \Pi_{QSym}$ [2, Proposition 6.1]. Now $\Pi_{QSym} = \Pi$, the peak algebra with basis given in (8), and $\Pi_{\mathcal{P}}$

contains the subalgebra of all Eulerian posets. Together, this gives another proof of Theorem 3.5.

The author and Aguiar are currently working to extend the theory of combinatorial Hopf algebras to the case of nonhomogeneous polynomial characters. One outcome is an alternate definition of the complete quasisymmetric function \tilde{F} defined in (9) and a new proof of Theorem 4.4.

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