

MATH 2210: SUPPLEMENTARY NOTES ON WEEK 2

ABSTRACT. In the following we review some important concepts in Sec 2.1 and 2.2.

Keywords: (1) Stereographic projection; (2) graphs of functions of multivariables; (3) continuity and uniform continuous.

Without specification, all numbers and symbols correspond to the textbook (Lax-Terrell 2016).

Part 1: Stereographic projection

Def (2.20 p77) (Stereographic projection): The unit sphere in three-dimensional space $\mathbb{R}^3$ is the set $S = \{(X, Y, Z) \in \mathbb{R}^3 \mid X^2 + Y^2 + Z^2 = 1\}$. Let $N = (0, 0, 1)$ be the “north pole”, Stereographic projection is a function $S : \mathbb{R}^2 \rightarrow S \setminus \{N\}$ which maps any point $B = (x, y, 0)$ on the plane $z = 0$ to a point A on the sphere. Here A is the intersection point of the line through B and N with the unit sphere. See the picture below ¹

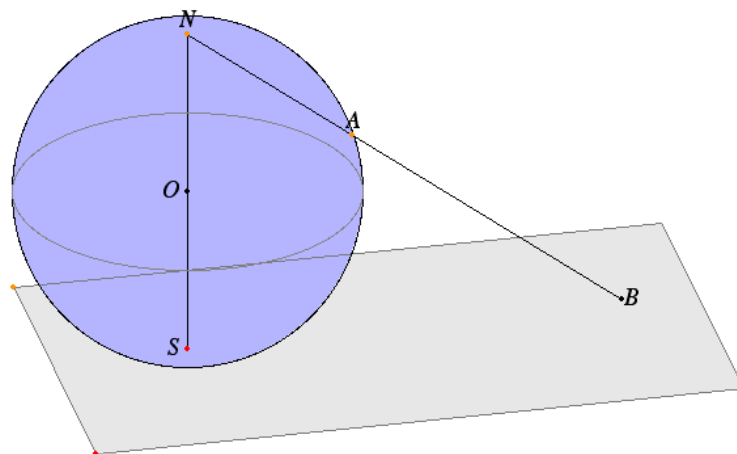


FIGURE 1. stereographic projection from north pole

In Lecture 3 we have shown:

Proposition (2.20 p77) $S : \mathbb{R}^2 \rightarrow S \setminus \{N\}$ has the formula:

$$S(x, y) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right)$$

It is not hard to show that S has an inverse function $F : S \setminus \{N\} \rightarrow \mathbb{R}^2$:

$$F(X, Y, Z) = \left(\frac{X}{1 - Z}, \frac{Y}{1 - Z} \right).$$

Next we state two remarkable properties of $F : S \setminus \{N\} \rightarrow \mathbb{R}^2$.

¹Blatantly copied from <http://aleph0.clarku.edu/~djoyce/java/compass/compass4.html>

Theorem 1: Any circle that does not pass through the north pole $(0, 0, 1)$ is mapped by F to a circle on the plane $z = 0$.

Intuition: It is easy to see the circle of latitude is indeed mapped by F to circles on the plane $z = 0$. In general, here is a picture: ²

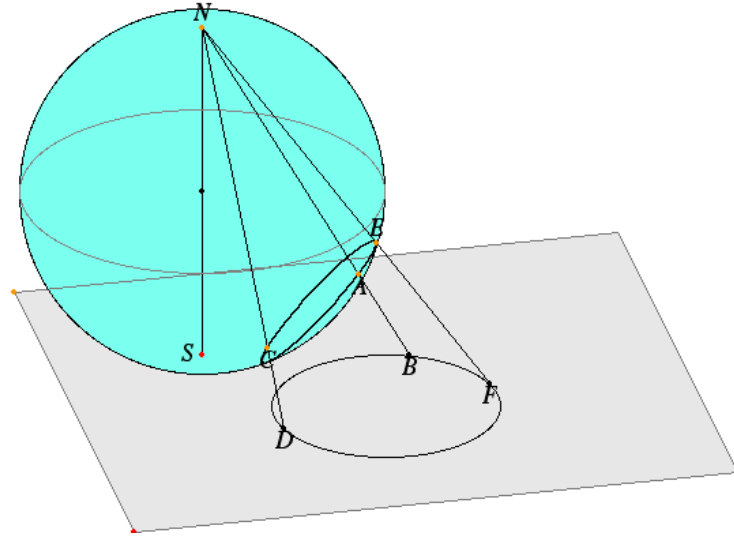


FIGURE 2. F maps a circle to a circle

Proof of Theorem 1 by an algebraic method: Any circle C on the unit sphere can be written as intersection of a plane in $\mathbb{R}^3$ and the sphere.

$$(1) \quad \begin{cases} X^2 + Y^2 + Z^2 = 1, \\ AX + BY + CZ + D = 0. \end{cases}$$

According to the formula of the components of the stereographic projection S are exactly $S(x, y) = (X, Y, Z)$, we have

$$(2) \quad (X, Y, Z) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

Plug (2) back to (1) we have

$$(3) \quad A \frac{2x}{x^2 + y^2 + 1} + B \frac{2y}{x^2 + y^2 + 1} + C \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} + D = 0.$$

Simplify it.

$$(4) \quad (C + D)x^2 + 2Ax + (C + D)y^2 + 2By - C + D = 0$$

The crucial assumption is that the original circle C does not pass the north pole N , which N plug into (1) implies $C + D \neq 0$. Now we will see (4) is a circle on the plane, just rewrite (4) into:

$$(5) \quad \left(x + \frac{A}{C + D}\right)^2 + \left(y + \frac{B}{C + D}\right)^2 = \frac{A^2 + B^2}{(C + D)^2} + \frac{C - D}{C + D}.$$

This finishes the proof of Theorem 1.

²Shame again! Copied from <http://aleph0.clarku.edu/~djoyce/java/compass/compass4.html>

Theorem 2: Stereographic projection $S : \mathbb{R}^2 \rightarrow S \setminus \{N\}$ and $F : S \setminus \{N\} \rightarrow \mathbb{R}^2$ preserves angles, in the sense that if two curves intersect at an angle θ on the sphere, so do their images curves on the plane $z = 0$ under F .³

Intuition: Think about two circles of longitude which meet at the south pole $(0, 0, -1)$, whose angle at south pole will be the difference of their longitudes. Under the map F , it becomes two lines passing through the origin $(0, 0, 0)$, meeting at the origin with the same angle!

Stereographic projection is useful because of Theorem 1 and Theorem 2. In general, it is not possible to map a portion of the sphere into the plane without introducing some distortion. Usually one cares about either the area or the angle. Stereographic projection is the latter. Angle-preserving map projections are important for navigation and it has an application in cartography. For example, one can use the map $F : S \setminus \{N\} \rightarrow \mathbb{R}^2$ to send the south hemisphere to the plane $z = 0$, here is an example:⁴

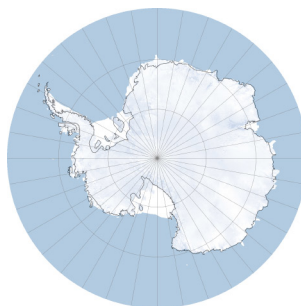


FIGURE 3. Map of the South Pole by stereographic projection

Also, in the above discussion we are using north pole for the reference point, the downside is that it is impossible to view maps near north pole. The following example use a reference point on the equator instead of north pole.⁵

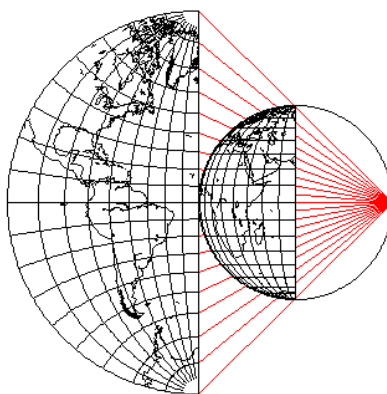


FIGURE 4. Stereographic projection from equator

³If you are interested, read here <http://www.ams.org/samplings/feature-column/fc-2014-02> for a geometric proof.

⁴Copied from <http://earthobservatory.nasa.gov/blogs/elegantfigures/2010/09/27/g-projector/>

⁵Copied from <http://www.quadibloc.com/maps/maz0202.htm>

Part 2: Examples of graphs of functions of several variables

We collect some examples of graphs of functions of several variables.

- (1) A helix given by $F_1 : [0, 4\pi] \subset \mathbb{R}^1 \rightarrow \mathbb{R}^3$ defined by $F(t) = (\cos t, \sin t, t)$.

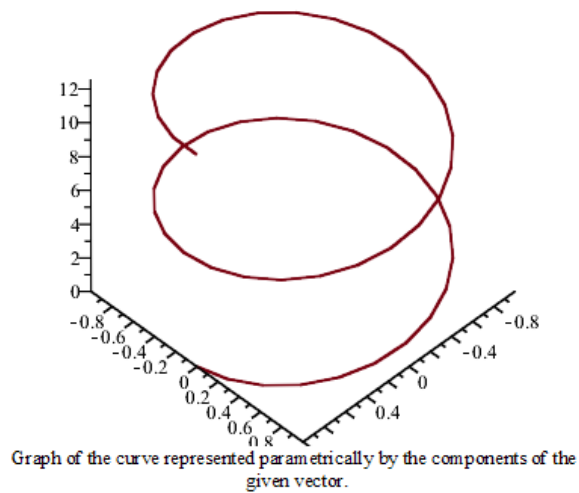


FIGURE 5. the graph of a helix

- (2) A spiral given by $F_2 : [0, 4\pi] \subset \mathbb{R}^1 \rightarrow \mathbb{R}^2$ defined by $F(t) = (t \cos t, t \sin t)$.

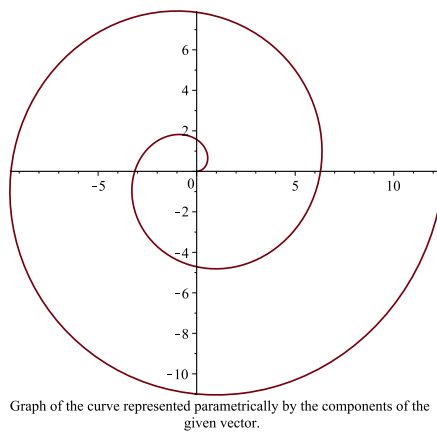
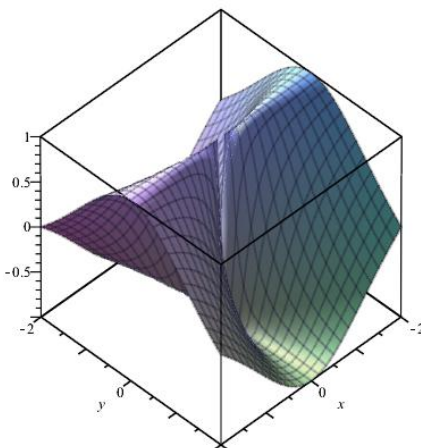
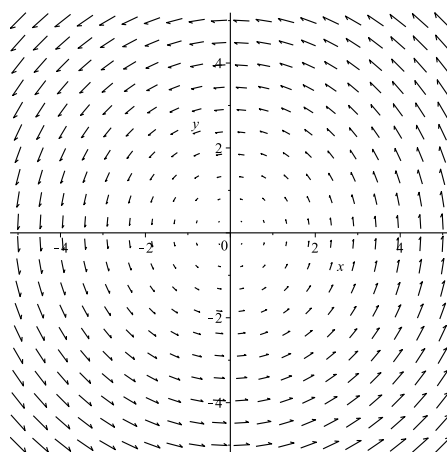


FIGURE 6. the graph of a sprial

- (3) (Example 2.8 p68) $F : \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}^1$ defined by $F(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$.

FIGURE 7. the graph of F on $[-2, -2] \times [-2, 2]$, but excluding $(0, 0)$

(4) (Example 2.71 p71) The plane vector field $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $F(x, y) = (-y, x)$. We may see in the graph that it models a rotation around the origin $(0, 0)$.

FIGURE 8. the graph of F on $[-5, -5] \times [-5, 5]$

Part 3: Continuity, discontinuous, uniform continuous

Def (2.6 p79) (Continuous functions from $D \subset \mathbb{R}^n \rightarrow \mathbb{R}^1$): A function $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^1$ is continuous at $X \in D$ if given any tolerance $\epsilon > 0$, there exists a precision δ which usually depends on X and ϵ such that if

$$\|X - Y\| < \delta \text{ and } Y \in D \quad \text{then} \quad |F(X) - F(Y)| < \epsilon$$

A function $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous at $X \in D$ if any i -th component of F , now as a function from $f^i : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^1$ is continuous at $X \in D$ in the above definition.

A function $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a continuous function on D if it is continuous at any point $X \in D$.

Def (2.16 p91) (Uniformly continuous function from $D \subset \mathbb{R}^n \rightarrow \mathbb{R}^1$): A function $F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^1$ is **uniformly continuous** on D if given any tolerance $\epsilon > 0$, there exists a

precision δ which only depends on ϵ such that if

$$\|X - Y\| < \delta \text{ for any given } X, Y \in D \quad \text{then} \quad |F(X) - F(Y)| < \epsilon$$

While uniformly continuous implies continuous, the inverse is only true after we impose extra assumptions on the domain of the function.

Th (2.12 p91) A continuous function $F : C \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ where C is a closed and bounded set is uniformly continuous on C

Now we would like to give an example to emphasize the difference of ‘continuous’ and ‘uniformly continuous’. In view of Theorem 2.12 above, we have to work on a set C which is either not closed or not bounded in order to find such examples.

Two examples on continuous but not uniformly continuous:

(1) $F : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ where $F(x, y) = x^2$. First F is continuous on $\mathbb{R}^2$ since it is product of two same linear functions $f(x, y) = x$, but why F is not uniformly continuous on $\mathbb{R}^2$?

The reason is that we can find two sequence of points $X_n, Y_n \in \mathbb{R}^2$ with $\|X_n - Y_n\| \rightarrow 0$ but $|F(X_n) - F(Y_n)| \geq \epsilon_0$ where ϵ_0 is a fixed constant. This is exactly the opposite of the definition of ‘uniformly continuous’! Why? Compare with Def (2.16 p91).

To make it work, just pick $\epsilon_0 = 1$ and $X_n = (n, n)$ and $Y_n = (n + \frac{1}{n}, n)$. Now check $\|X_n - Y_n\| = \frac{1}{n} \rightarrow 0$ but $|F(X_n) - F(Y_n)| \geq 2 > \epsilon_0$.

(2) $F : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^1$ where $F(x, y) = \frac{1}{1-x^2-y^2}$ where $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$. Again F is continuous on $\mathbb{R}^2$ since it is a fraction and the denominator $1 - x^2 - y^2$ is continuous (why?), but F is not uniformly continuous on D .

Why? By the similar reasoning as above, It suffices to find two sequence of points $X_n, Y_n \in \mathbb{R}^2$ with $\|X_n - Y_n\| \rightarrow 0$ but $|F(X_n) - F(Y_n)| \geq \epsilon_0$ where ϵ_0 is a fixed constant.

We pick $X_n = (1 - \frac{1}{n}, 0)$ and $Y_n = (1 - \frac{2}{n}, 0)$ for $n \geq 3$. Note that both $X_n, Y_n \in D$ and $\|X_n - Y_n\| = \frac{1}{n} \rightarrow 0$. Now let us check

$$\begin{aligned} |F(X_n) - F(Y_n)| &= \frac{1}{1 - (1 - \frac{1}{n})^2} - \frac{1}{1 - (1 - \frac{2}{n})^2} \\ &= \frac{2n^3 - 3n^2}{(2n - 1)(4n - 4)} \end{aligned}$$

Note that $\lim_{n \rightarrow \infty} \frac{2n^3 - 3n^2}{(2n - 1)(4n - 4)} \rightarrow +\infty$. Therefore, pick some $\epsilon_0 = 10$, then we find $|F(X_n) - F(Y_n)| \geq \epsilon_0$ as n increases.