Math 453
Solutions to Assignment 10
November 23, 2004

Exercise 5 from Section 30, page 194

(a) Let D be a countable dense subset of the metrizable space X. I claim that

$$\mathscr{B} = \{B(x, 1/n) \mid x \in D \text{ and } n \in \mathbb{Z}^+\}$$

is a countable basis. First, $\mathscr{B}$ is countable, since both D and $\mathbb{Z}^+$ are. To show that it is a basis for the topology on X, let U be any open set, and let $y \in U$. Then there is an open ball, $B(y, \varepsilon)$, around y that is contained in U. Choose n to be larger that $2/\varepsilon$, and pick an $x \in D$ such that $x \in B(y, 1/n)$. This gives $y \in B(x, 1/n) \subset B(y, \varepsilon) \subset U$. Hence $\mathscr{B}$ is a basis.

(b) Let X be a metrizable Lindelöf space. For each $n \in \mathbb{Z}^+$, the collection of open sets $\{B(x, 1/n) \mid x \in X\}$ covers X. Pick a countable subcover and call it $\mathscr{B}_n$. Then the collection $\mathscr{B} = \bigcup_{n=1}^{\infty} \mathscr{B}_n$ is a countable basis. Countability is immediate, and the proof that it is a basis is analogous to that in part (a). □

Exercise 6 from Section 30, page 194

We know from Example 3 (p. 192) that $\mathbb{R}_\ell$ is not second countable, but has a countable dense subset. Thus part (a) above implies that $\mathbb{R}_\ell$ cannot be metrizable.

Suppose I_o^2 were metrizable. Since I_o^2 is also compact (and hence Lindelöf), part (b) above implies that it is second countable. This in turn shows that the subspace $A = I \times (0, 1)$ is second countable, and hence Lindelöf. However, Example 5 (p. 193) shows that A is not Lindelöf, so I_o^2 cannot be metrizable. □

Exercise 12 from Section 30, page 194

I'll show the case for second countability; the only difference for first countability is to start with a countable basis at a point instead. Suppose $\mathscr{B}$ is a countable basis for X. Since f is an open map,

$$\mathscr{B}' = \{f(B) \mid B \in \mathscr{B}\}$$

is a collection of open sets in $f(X)$. To show that it is a basis for $f(X)$, let V be an open set in $f(X)$ and let $y \in V$. Pick any x for which $y = f(x)$; then $x \in f^{-1}(V)$, so there is a basis element $B \in \mathscr{B}$ such that $x \in B \subset f^{-1}(V)$. This gives $y \in f(B) \subset V$, so $\mathscr{B}'$ is a basis. Since it is also countable, $f(X)$ is second countable. □

Exercise 3 from Section 31, page 199

Let X be a simply ordered set with the order topology. Let A be a closed subset of X and x be an element of X not in A. $X - A$ is open and $x \in X - A$, so there is an open interval (a, b)

such that $x \in (a,b) \subset X - A$. (If x is the largest or smallest element of X, use $(a,x]$ and $[x,b)$ instead, and make the obvious changes in what follows.) Pick any $c \in (a,x)$ if such an element exists, or $c = x$ otherwise; similarly, pick $d \in (x,b)$ or set $d = x$ if $(x,b) = \emptyset$. Then $(-\infty, c) \cup (d, \infty)$ is an open set that contains A, and (c,d) (or (a,d) if $c = x$, or (c,b) if $d = x$, or (a,b) if $c = d = x$) is an open set containing x. They are clearly disjoint, so X is regular. □

Exercise 7 from Section 31, page 199

(a) First we show that given any $y \in Y$ and open U that contains $p^{-1}(\{y\})$, there is a neighborhood W of y such that $p^{-1}(W) \subset U$. Since U is open, $X - U$ is closed, so $p(X-U)$ is closed. Set $W = Y - p(X-U)$. Then W is open, $y \in W$ and $p^{-1}(W) \subset U$.

Now suppose X is Hausdorff and let y_1 and y_2 be distinct points of Y. p is a perfect map, so $p^{-1}(\{y_i\})$, $i = 1, 2$ are disjoint compact subsets of X; find disjoint open sets U_i that contains $p^{-1}(\{y_i\})$ using the result of Exercise 26.5. Finally, choose neighborhoods W_i of y_i such that $p^{-1}(W_i) \subset U_i$. Since U_1 and U_2 are disjoint, so are W_1 and W_2. Hence Y is Hausdorff.

(b) Let $y \in Y$ and U be a neighborhood of y. We need to find a neighborhood V of y such that $\overline{V} \subset U$.

Now p is continuous, so $p^{-1}(U)$ is an open set containing $p^{-1}(\{y\})$. Using regularity of X, find, for each $x \in p^{-1}(\{y\})$, an open set W_x such that $x \in W_x$ and $\overline{W}_x \subset p^{-1}(U)$. This gives an open cover $\{W_x \mid x \in X\}$ of $p^{-1}(\{y\})$, from which we can extract a finite subcover $\{W_{x_1}, \dots, W_{x_n}\}$. Since $W_{x_1} \cup \cdots \cup W_{x_n}$ is an open set containing $p^{-1}(\{y\})$, we can find a neighborhood V of y such that $p^{-1}(V) \subset W_{x_1} \cup \cdots \cup W_{x_n}$. It remains to show that $\overline{V} \subset U$.

$\overline{W}_{x_1} \cup \cdots \cup \overline{W}_{x_n}$ is a closed set that contains $p^{-1}(V)$, so $\overline{p^{-1}(V)} \subset \overline{W}_{x_1} \cup \cdots \cup \overline{W}_{x_n}$. Also, by construction, $\overline{W}_{x_1} \cup \cdots \cup \overline{W}_{x_n} \subset p^{-1}(U)$. Hence $\overline{p^{-1}(V)} \subset p^{-1}(U)$, giving $\overline{V} \subset p(\overline{p^{-1}(V)}) \subset U$.

(c) Let $y \in Y$ and consider $p^{-1}(\{y\})$. For each $x \in p^{-1}(\{y\})$, there is a compact subset C_x of X that contains a neighborhood U_x of x. $\{U_x \mid x \in p^{-1}(\{y\})\}$ covers $p^{-1}(\{y\})$, so via compactness, we obtain a finite subcover, say $\{U_{x_1}, \dots, U_{x_n}\}$. A finite union of compact sets is compact, so $C = C_{x_1} \cup \cdots \cup C_{x_n}$ is a compact and contains $U = U_{x_1} \cup \cdots \cup U_{x_n}$. Using what we first proved in part (a), there is a neighborhood W of y such that $p^{-1}(W) \subset U$. Now $\overline{p^{-1}(W)} \subset C$, so it is compact, and $p(\overline{p^{-1}(W)})$ is a compact subset of Y. It contains the neighborhood W of y, so Y is locally compact.

(d) Suppose X is second countable, with countable basis $\mathscr{B}$. For each finite subset J of $\mathscr{B}$, let U_J be the union of all sets of the form $p^{-1}(W)$, for W open in Y, that are contained in the union of elements of J. First observe that $p(p^{-1}(W)) = W$ since p is surjective, so $p(U_J)$ is open, being a union of open sets. We will show that $\mathscr{C} = \{p(U_J) \mid J \text{ is a finite subset of } \mathscr{B}\}$ is a countable basis for Y. Countability is easy, since the collection of finite subsets of a countable set is countable.

To show that $\mathscr{C}$ is a basis for Y, let V be an open set in Y and let $y \in V$. Then $p^{-1}(V)$ is an open set that contains $p^{-1}(\{y\})$, so for each $x \in p^{-1}(\{y\})$, there is an element $B_x \in \mathscr{B}$ such that $x \in B_x \subset p^{-1}(V)$. Using compactness of $p^{-1}(\{y\})$, find a finite subcover $J = \{B_{x_1}, \ldots, B_{x_n}\}$. Now $B_{x_1} \cup \cdots \cup B_{x_n}$ is an open set containing $p^{-1}(\{y\})$, so there is a neighborhood W of y such that $p^{-1}(W) \subset B_{x_1} \cup \cdots \cup B_{x_n}$. Hence U_J contains $p^{-1}(\{y\})$, or $y \in p(U_J)$. Furthermore, $p(U_J) \subset V$, so $\mathscr{C}$ is a countable basis for Y. □

Exercise 1 from Section 32, page 205

Let X be a normal space and let $Y \subset X$ be a closed subspace. Suppose A and B are disjoint closed subsets of Y. Then A and B are closed in X, since Y is closed. Thus we can find disjoint open (in X) sets U and V such that $A \subset U$ and $B \subset V$. Then $U \cap Y$ and $V \cap Y$ are the required disjoint open sets containing A and B, respectively. Hence Y is normal. □

Exercise 4 from Section 32, page 205

Let X be a regular Lindelöf space. Let A and B be disjoint closed subsets of X. B is closed, so each point a of A has a neighborhood W_a not intersecting B; using regularity, choose a neighborhood U_a of a such that $\overline{U_a} \subset W_a$. This gives $\overline{U_a} \cap B = \varnothing$. The collection $\{U_a \mid a \in A\}$ covers A, and A, being a closed subspace of a Lindelöf space, is Lindelöf (the proof is the same as that for showing that a closed subspace of a compact space is compact), so we can find a countable subcover, say $\{U_1, U_2, \ldots\}$.

Similarly, choose a countable collection $\{V_n\}$ of open sets covering B such that $\overline{V_n} \cap A = \varnothing$. The rest of the proof is an exact copy of that of Theorem 32.1, starting from the second paragraph. □