Noncyclic Covers of Knot Complements

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ABSTRACT

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Let K be a knot in S^3 and $M = S^3 - K$ its complement. The finite cyclic covers of M are well known. However, much less is known about noncyclic covers of M. Hempel has shown that the fundamental groups of Haken 3-manifolds are residually finite. It follows that if K is a nontrivial knot then M has a noncyclic cover with finitely many sheets. We give an explicit function, $\Phi : \mathbb{N} \to \mathbb{N}$, such that if K is a nontrivial knot with a diagram with c crossings then M has a noncyclic cover with at most $\Phi(c)$ sheets.

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1 Introduction

Let K be a nontrivial knot in S^3. Let $M = S^3 - N(K)$ be the complement of an open regular neighborhood of K. It is well known that for each positive integer, k, M has a unique cyclic cover with k sheets arising from the map $\pi_1(M) \to H_1(M) \cong \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$. However, much less is known about the noncyclic covers of knot complements. In [Hem87] Hempel establishes that fundamental groups of Haken 3-manifolds are residually finite. This shows in particular that the fundamental groups of knot complements are residually finite. Thus for any nontrivial element, g, of the commutator of $\pi_1(M)$ there must be a nontrivial normal subgroup of finite index in $\pi_1(M)$ not containing g. This shows that knot complements must have infinitely many finite, nonabelian covers. The goal of this exposition is to give an explicit function $\Phi(c)$ such that if K is a nontrivial knot with a diagram with c crossings and $M = S^3 - N(K)$ is its complement then M has a noncyclic cover with at most $\Phi(c)$ sheets.

As motivation for such a result, one should notice that this gives an algorithm to establish that a knot is nontrivial. If one starts with a knot with c crossings and systematically creates all covers of the complement with $\Phi(c)$ or less sheets then if a noncyclic cover is found, the knot is nontrivial. If no such cover is found, the knot is trivial. This algorithm is different from most existing algorithms to detect knottedness in that it searches for a proof that the knot is nontrivial instead of a proof that the knot is trivial.

2 Main Result

Set

$$A(n) = \frac{(n^2 - n + 1)!}{n^2[(n-1)!]^n}, \tag{1}$$

$$B(n) = \frac{n^3 - n^2}{n^2 - n + 1}, \tag{2}$$

and

$$\begin{aligned} D(n) &= \exp\Big[2\left(72n^2+12n\right)^{n2^{n-1}}\left((12n3^n-1)\log 2 + A(n)(18n)\right. \\ &\qquad \left.\cdot\left(n2^{3n+1}+2^{6n+2}+B(n)\log 2\right)\left(72n^2 3^{2n}+18n3^n-1\right)\right)\Big]. \end{aligned} \tag{3}$$

The main result is as follows:

Theorem 1. *Let K be a nontrivial knot in S^3 and $M = S^3 - K$ its complement. Suppose K has a diagram with c crossings. Then M has a noncyclic cover with at most $\Phi(c) = (2D(100c))^{8c\left(1080000c^2 24^{100c}+2\right)}/D(100c)^{16c-2}$ sheets.*

The proof of Theorem 1 will proceed using Thurston's geometrization for knot complements. Let K and M be as in the statement of the theorem. Then the JSJ decomposition of M cuts M along essential tori $T_1, T_2, \cdots, T_r$ into spaces $M_0, M_1, \cdots, M_r$ where either M_i is Seifert fibered or $M_i - \partial M_i$ has a complete hyperbolic structure. As shown in [NS97], no annuli are needed because M has Euler characteristic 0. Note that this decomposition is the same as the satellite knot decomposition of M.

2.1 Standard spines and ideal triangulations

For our purposes an *ideal triangulation* of a 3-manifold, M, will be a simplicial complex, $\mathcal{T}$, satisfying some further conditions. The complex, $\mathcal{T}$, must be a union of a finite number of 3-simplices with pairs of faces identified. In fact, we insist that there are no unidentified "free" faces. Identification of different faces of the same tetrahedron will be allowed. For $\mathcal{T}$ to be an ideal triangulation of the 3-manifold, M, we require $\mathcal{T}$ minus its vertices to be homeomorphic to $M - \partial M$. We will write $\mathcal{T} = \bigcup_{i=1}^{n} \sigma_i$ to indicate that $\mathcal{T}$ is an ideal triangulation with the n ideal tetrahedra, $\sigma_1, \sigma_2, \ldots, \sigma_n$. For our purposes, the links of the vertices in our ideal triangulations will always be tori, and all 3-manifolds and their ideal triangulations will be orientable.

When dealing with the geometric pieces of M it will be convenient to have a bound on the number of tetrahedra needed to triangulate them. In working with ideal triangulations it is often helpful to be familiar with the dual notion of standard spines. As in [Cas65] a *spine* is simply a 2-complex. The *singular 1-skeleton* of a spine is the set of points which do not have neighborhoods homeomorphic to open disks. The *singular vertices* of a spine are the points of the singular 1-skeleton which do not have neighborhoods in the singular 1-skeleton homeomorphic to open intervals. Let C be a spine, C_1 be the singular 1-skeleton of C, and C_0 be the singular vertices of C. The spine C will be a *standard spine* if it satisfies three conditions. Firstly, C must satisfy the *neighborhood condition*. That is, every point of C must have a neighborhood homeomorphic to one of the three 2-complexes pictured in Figure 1.

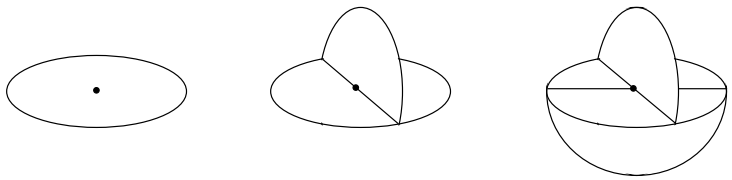


Figure 1: The three possible neighborhoods in a standard spine

Secondly, $C - C_1$ must be a union of countably many disjoint 2-disks. Thirdly, we require that $C_1 - C_0$ be a union of countably many disjoint arcs. The complex, C, is a *spine* (resp. *standard spine*) *of a 3-manifold* N if $C \subset N$ is a spine (resp. standard spine) and N collapses to C. An important property of a standard spines is that if C is a standard spine of N then if C is embedded in any 3-manifold then N is homeomorphic to a regular neighborhood of C in that manifold.

In [TV92] and [Pet97] it is mentioned that standard spines are dual to ideal triangulations (see Figure 2). For every ideal triangulation of a 3-manifold there is a dual standard spine, and for every standard spine there is a dual ideal triangulation. Thus we see that a standard spine carries the same information as an ideal trian-

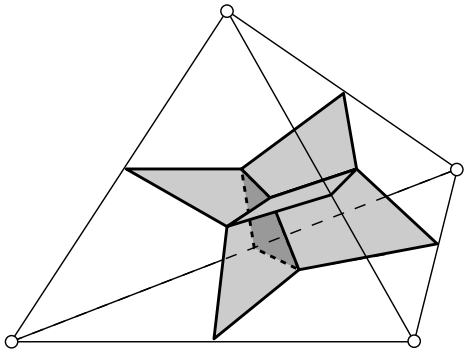


Figure 2: An ideal tetrahedron and its dual spine

gulation. Moreover, the number of singular vertices in a standard spine will be the number of ideal tetrahedra in the dual triangulation. We will exploit this duality a number of times.

2.2 Triangulating a knot complement

A preliminary step in our exposition will be to relate the number of crossings in a knot diagram to the number of ideal tetrahedra needed to triangulate the complement of the knot. It is noted in [Pet97] that the number of ideal tetrahedra needed is at most linear in the number of crossings in a projection. Here we give an explicit relationship.

Lemma 1. *Let K be a knot in S^3 with a diagram with $c > 0$ crossings. Then the complement of K has an ideal triangulation with less than $4c$ ideal tetrahedra.*

Proof. Let K be a knot in S^3 with a diagram with c crossings. We will produce a standard spine based on this projection which will have less than $4c$ singular vertices. This spine will be dual to an ideal triangulation of the knot complement with less than $4c$ ideal tetrahedra.

We will assume that the projection of K is a series of arcs in the equatorial 2-sphere, S, in S^3. Create a spine, C', as shown in Figure 3. Note that C' has exactly $4c$ singular vertices. For convenience we will assume that K is the core curve of the

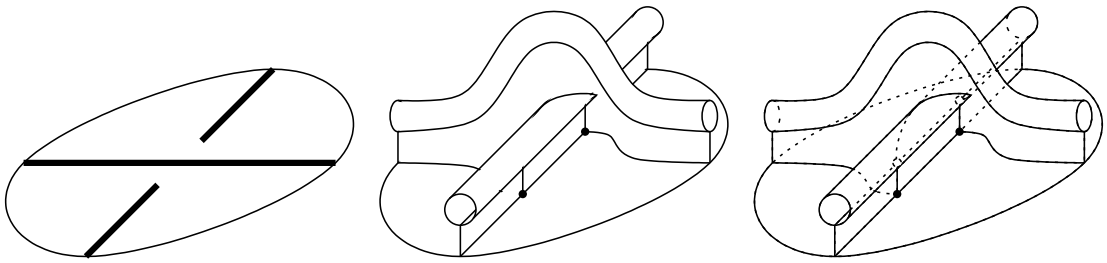


Figure 3: A knot projection and two views of the corresponding spine C'

"tubes" in Figure 3. It should be clear that C' satisfies the neighborhood condition. As long as our projection contains at least one crossing, the complement in C' of its singular 1-skeleton will be a union of disks. However, C' is not a spine of $M = S^3 - K$ since two components of $M - C'$ are separated from ∂M.

We now modify C' to produce a standard spine of M. Let C'_1 be the singular 1-skeleton of C'. Note that C' divides S^3 into 3 components. Two are homeomorphic to open 3-balls, and one is an open regular neighborhood of K which we will denote by N. Choose a disk, D_1, in $\partial N - C'_1$. The closure of D_1 in S^3 contains a maximal open arc, λ, such that $D_1 \cup \lambda$ is an open annulus. Let D_2 be the open disk in $C' - (C'_1 \cup D_1)$ whose closure contains λ. Set $C'' = C' - (D_1 \cup \lambda \cup D_2)$, and let C''_1 be its singular 1-skeleton. Choose an open disk D_3 in $S - C''_1$. Let $C = C'' - D_3$.

The spine, C, is, in fact, a standard spine for M. To prove this we first show that M collapses to C. It should be apparent than $\bar{N} - K$ collapses to ∂N. The removal of $(D_1 \cup \lambda \cup D_2)$ from C' joins $N - K$ to a region homeomorphic to an open solid torus along an open annulus. The closure in M of this new region collapses to it's boundary. Finally, the removal of D_3 joins an open 3-ball to this region along an open disk. This joined region is $M - C$ and we see that its closure in M collapses to C.

Now we must show that C is a standard spine. The modification from C' to

C preserves the neighborhood condition. Let C_1 be the singular 1-skeleton of C. We will see that $C'' - C_1''$ is a disjoint union of open disks and then that $C - C_1$ is as well.

First consider the effect of removing $(D_1 \cup \lambda \cup D_2)$ from C' to get C''. The equatorial 2-sphere, S, intersects C_1' in a graph which agrees with the actual knot projection except near crossings. In fact, C_1' and the knot projection cut S into nearly identical 2-disks. In S, the removal of $(D_1 \cup \lambda \cup D_2)$ eliminates the arc in C_1' corresponding to the projection of λ onto S. This has the effect of joining certain disks in $S - C_1'$. These joined regions will all be disks because the knot diagram will still be connected after the removal of any one over-arc. The other change after the removal of $(D_1 \cup \lambda \cup D_2)$ is that the vertical walls at both ends of λ will be joined to the pieces of the tunnel on the other side of the wall from λ. This amounts to gluing a disk to a disk along an arc in their boundaries. The results are still disks. Hence, $C'' - C_1''$ is a disjoint union of open disks.

Now consider the effect of removing D_3 from C'' to get C. Here each vertical wall surrounding D_3 will be joined to the disk in S on the other side of the wall from D_3. These disks are joined along single arcs in their boundary; therefore, they glue together to form disks. Consequently, $C - C_1$ is a union of open disks.

Let C_0 be the set of singular vertices of C. I claim that $C_1 - C_0$ must be a disjoint union of open arcs. If not then $C_1 - C_0$ contains an S^1. The spine, C, is connected and $C - C_1$ is composed entirely of disks so each disk must, in fact, have this S^1 as its boundary. This is impossible because the only such spine satisfying the neighborhood condition is composed of three disks glued along their boundary. This is not a spine of the complement of a knot in the 3-sphere or the complement of a knot in a solid torus. Consequently, C is a standard spine of $M - K$.

The standard spine, C, has strictly fewer singular vertices than C', so C has less than $4c$ singular vertices. It follows that there is a dual ideal triangulation of M with less than $4c$ ideal tetrahedra. □

2.3 Triangulating pieces of the JSJ decomposition

Now that we have bounded the number of ideal tetrahedra needed to triangulate M, we can consider the number of tetrahedra needed to triangulate the geometric pieces of M.

Lemma 2. *Let K be a knot in S^3 and $M = S^3 - K$ its complement. Suppose M can be triangulated with t ideal tetrahedra. Also suppose that embedded, disjoint tori, $T_1, T_2, \cdots, T_r$, give the JSJ decomposition of M, and $M_0, M_1, \cdots, M_r$ are the connected components after cutting. Then the M_i's have ideal triangulations with t_i ideal tetrahedra each so that $\sum_{i=0}^{r} t_i \leq 25t$.*

Proof. Let K and M be as in the statement of the lemma. By assumption M has an ideal triangulation $\mathcal{T} = \bigcup_{i=1}^{t} \sigma_i$ with t ideal tetrahedra. We may choose our tori $T_1, T_2, \cdots, T_r$ so that their union S is a normal surface with respect to $\mathcal{T}$. For each $i, 1 \leq i \leq t$, S cuts σ_i into pieces with four basic types (see Figures 4 and 5):

(a) Pieces whose closure intersects S in two triangles.

(b) Pieces whose closure intersects S in two quadrilaterals.

(c) Pieces whose closure intersects S in two triangles and one quadrilateral.

(d) Pieces whose closure intersects S in four triangles.

Of course there will also be pieces whose closures in σ_i will be incident with the corners of σ_i, but we will put these pieces in categories (a) - (d) based on how they look when the corners of σ_i are cut off by triangles.

We will now construct a standard spine of $M - S$. Consider the ith tetrahedron in the ideal triangulation of M and its intersection with the surface S. For each region of type (a) and (b) place a triangular or quadrilateral disk in its center parallel to the faces incident with S as shown in Figure 4. For each region of type (c) and (d) place

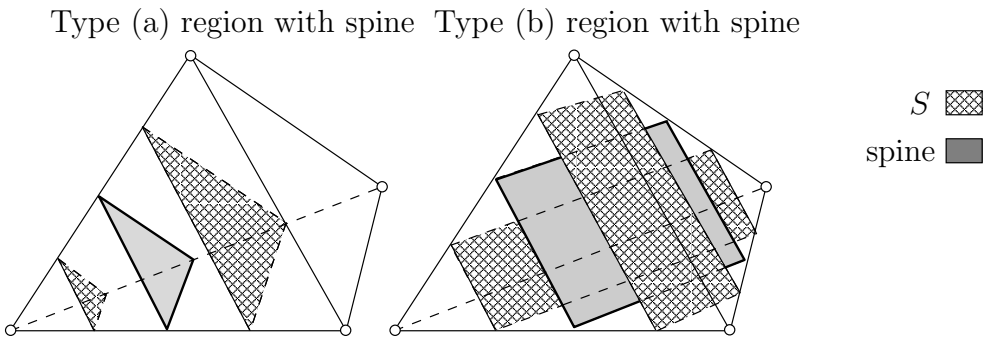


Figure 4: Type (a) and (b) spines

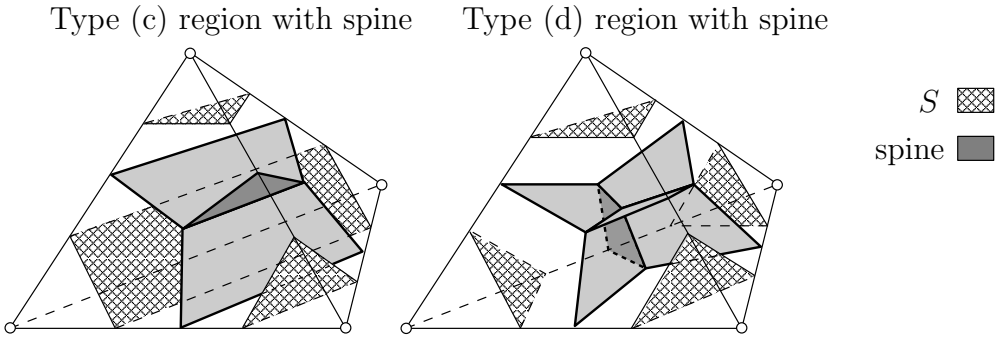


Figure 5: Type (c) and (d) spines

a 2-complex as in Figure 5. Let $D_i \subset \sigma_i$ be the union of all of these spine pieces. One sees immediately that $C' = \bigcup_{i=1}^{t} D_i$ is a spine of $M - S$ and that it satisfies the neighborhood condition. The spine, C', has a singular vertex for each region of type (d). Clearly each tetrahedron of $\mathcal{T}$ contains at most one region of type (d). Thus C' has at most t vertices. Note that each connected component of C' has a nonempty singular 1-skeleton, for if there were a component, A, of C' with empty singular 1-skeleton then the component of $M - S$ containing A would be homeomorphic to the torus crossed with the open unit interval.

Let C'_1 be the singular 1-skeleton of C'. Although C' satisfies the neighborhood condition, C' will not in general be a standard spine since $C' - C'_1$ may not be a union of disks. A component of $C' - C'_1$ must have genus 1 or 0 since the boundary components of $M - S$ are tori. Thus if a component of $C' - C'_1$ has b boundary

components it may be cut into disks with $b+1$ or less arcs. For each of these arcs, γ, modify C' as in Figure 6 (This is possible since each connected component of C' has a nonempty singular 1-skeleton). Let C be the modified spine. The modification in

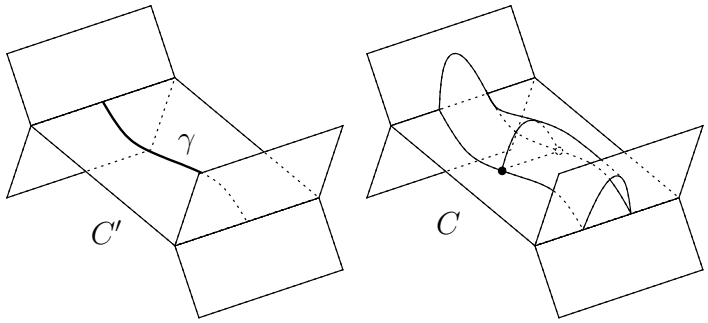


Figure 6: Modifying C' to get C

Figure 6 takes one spine to another [Pie88] so C is a spine of $M - S$. The spine, C, satisfies the neighborhood condition. Let C_1 be the singular 1-skeleton of C and C_0 be the singular vertices of C_1. By construction, $C - C_1$ is a disjoint union of disks. I claim that $C_1 - C_0$ must be a disjoint union of arcs. If not then $C_1 - C_0$ contains an S^1. Since $C - C_1$ is composed entirely of disks, each disk in the component of C containing this S^1 must have this S^1 as its boundary. This is impossible because, as mentioned before, the only such spine satisfying the neighborhood condition is three disks glued along their boundary. This is not a spine of the complement of a knot in the 3-sphere or the complement of a knot in a solid torus. Consequently, C is a standard spine of $M - S$.

Let us now count the singular vertices of C. The total number of boundary components of $C' - C'_1$ is less than or equal to 3 times the number of regions of type (c) plus 6 times the number of regions of type (d). The number of components of $C' - C'_1$ is bounded by that same number. Let $\mathcal{A}$ be the set of connected components of $C' - C'_1$, and for $A \in \mathcal{A}$ let b_A be the number of boundary components of A. If a

is the number of arcs needed to cut $C' - C'_1$ into disks we have:

$$\begin{aligned} a &= \sum_{\substack{A\in\mathcal{A}\\ \mathrm{g}(A)=1}} (b_A+1) + \sum_{\substack{A\in\mathcal{A}\\ \mathrm{g}(A)=0}} (b_A-1) \\ &= \sum_{A\in\mathcal{A}} b_A + \sum_{\substack{A\in\mathcal{A}\\ \mathrm{g}(A)=1}} 1 - \sum_{\substack{A\in\mathcal{A}\\ \mathrm{g}(A)=0}} 1 \qquad (4) \\ &\leq 6t + |\mathcal{A}| \\ &\leq 12t \end{aligned}$$

Changing C' to C introduces 2 singular vertices for each cutting arc so C has at most $2a \leq 2 \cdot 12t = 24t$ more singular vertices than C'. As mentioned above, C' has at most t vertices so C has at most $25t$ vertices.

The standard spine, C, is dual to an ideal triangulation of $M - S$ with the same number of ideal tetrahedra as singular vertices of C. This shows that $M - S$ can be triangulated with $25t$ or less tetrahedra. □

In proof of Lemma 2 we saw that for each i, $M_i \cap C'$ must have nonempty singular 1-skeleton. The singular 1-skeleton of C' can have, at most, 2 components for every tetrahedron of $\mathcal{T}$ so we get the following corollary:

Corollary 1. *Let K be a knot in S^3 and $M = S^3 - K$ its complement. Suppose M can be triangulated with t ideal tetrahedra. Also suppose that the embedded, disjoint tori $T_1, T_2, \cdots, T_r$ give the JSJ decomposition of M. Then $r < 2t$.*

2.4 Decomposition of a knot complement

We can now resume our discussion of the decomposition of M into geometric pieces. View the cutting tori $T_1, T_2, \cdots, T_r$ as sitting inside M and hence S^3. The Solid Torus Theorem says that a torus in S^3 bounds a solid torus on at least one side (see [Rol90] pg. 107). Each cutting torus, T_k, divides S^3 into a piece which contains the knot K and a piece which does contain the knot. If the component of $S^3 - T_k$ which did not

contain K were a torus then T_k would have a compressing disk in M contradicting that T_k is essential. Consequently, T_k bounds a solid torus, V_k, containing K. We may order the cutting tori as follows: if T_k and T_j are two cutting tori with $j \neq k$ then set $T_k > T_j$ if $V_k \subset V_j$. Note that $V_k \cap V_j$ always contains K and so must be nonempty. If neither $V_k \subset V_j$ nor $V_k \subset V_j$ then $\partial V_k = T_k$ and $\partial V_j = T_j$ must intersect contradicting the disjointness of the cutting tori. Hence, we must have either $T_k > T_j$ or $T_j > T_k$. Renumber the cutting tori so that $T_1 < T_2 < \cdots < T_r$. Let M_0 be the piece of $\bigcup_{i=1}^{r} T_i$ which has one boundary component. For $i \geq 1$, M_i will have two boundary components. Thus we get M as a graph product of CW complexes based on the graph in Figure 7.

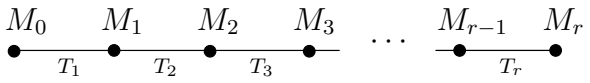


Figure 7: M as a graph product

In order to create a covering space of M, we will produce a compatible collection $\{\widetilde{M}_i\}$ of finite covers of each of the M_i's and assemble them into a finite cover, $\widetilde{M}$, of M following [Hem87] (see Figures 8 and 9). More specifically, we will choose a prime, p, and let $P \trianglelefteq \mathbb{Z} \times \mathbb{Z}$ be the characteristic subgroup generated by $(p, 0)$ and $(0, p)$. For each torus boundary component, T_{i-1} and T_i, of M_i we will let $\widetilde{T}_{i-1}$ and $\widetilde{T}_i$ be the covers associated to the subgroups of $\pi_1(T_{i-1})$ and $\pi_1(T_i)$ corresponding to P. We will then produce a finite cover, $\widetilde{M}_i$, of M_i all of whose boundary components will be equivalent to either $\widetilde{T}_{i-1}$ or $\widetilde{T}_i$. Finally we will assemble copies of these $\widetilde{M}_i$'s to get a cover of M. For each M_i the challenge will be to discover for which primes, p, we will be able to produce such a cover and then to bound the number of sheets in that cover.

3 Hyperbolic Pieces

3.1 Mahler measure and height

In order to address the case in which M_i is a hyperbolic manifold we will use the number of tetrahedra in an ideal triangulation of M_i to limit certain quantities related to a representation of $\pi_1(M_i)$ into the group isometries of hyperbolic 3-space. In the process we will encounter certain polynomial equations and algebraic numbers. For numerous reasons the most natural notions of complexity for polynomials and algebraic numbers are given by the Mahler measure and height, respectively. We define these notions here.

Let $P = P(X_1, X_2, \cdots, X_n)$ be a polynomial with complex coefficients. As in [MŞ99] the *Mahler measure*, $M(P)$, is given by

$$M(P) = \exp\left(\int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i t_1}, \cdots, e^{2\pi i t_n})|\, dt_1 \cdots dt_n\right). \tag{5}$$

From this formula one may derive

$$M(PQ) = M(P)M(Q). \tag{6}$$

As mentioned above, the Mahler measure of a polynomial will be a measure of its complexity. Two other notions of complexity which may at first seem more natural are the length and quadratic norm. If $P(X_1, \ldots, X_n) = \sum a_{j_1 \cdots j_n} X_1^{j_1} \cdots X_n^{j_n}$ is a polynomial with complex coefficients, then the *length* of P is

$$L(P) = \sum |a_{j_1 \cdots j_n}|, \tag{7}$$

and the *quadratic norm* of P is

$$\|P\| = \sqrt{\sum |a_{j_1 \cdots j_n}|^2}. \tag{8}$$

The following two lemmas from [MŞ99] relate these three notions. Lemma 2.1.7 of [MŞ99] is as follows:

Lemma 3. *If $P \in \mathbb{C}[X_1, \ldots, X_n]$, then $M(P) \leq \|P\|$.*

Lemma 2.1.9 of [MŞ99] relates the Mahler measure and length of a one-variable polynomial:

Lemma 4. *Let $P(X) = c_0 + c_1 X + \cdots c_m X^m \in \mathbb{C}[X]$ be a polynomial in one variable. Then*

$$|c_i| \leq \binom{m}{i} M(P).$$

In particular $|c_0|, |c_m| \leq M(P)$, and $L(P) \leq 2^m M(P)$.

Let $\alpha \in \mathbb{C}$ be an algebraic number of degree m and let $P(X)$ be its minimal polynomial over $\mathbb{Z}$. Define the measure $M(\alpha)$ of α to be

$$M(\alpha) = M(P). \tag{9}$$

Closely related to the measure of α is its *absolute multiplicative height*, $H(\alpha)$, given by the equation

$$H(\alpha) = M(\alpha)^{1/m}. \tag{10}$$

At times it is more convenient to consider the *absolute logarithmic height*, $h(\alpha)$, of an algebraic number α given by

$$h(\alpha) = \log H(\alpha) = \frac{1}{m} \log M(\alpha). \tag{11}$$

Let α and β be algebraic numbers. We have the following facts found in [Sch99] Lemma 2A:

$$H(\alpha\beta) \leq H(\alpha)H(\beta). \tag{12}$$

$$H(\alpha + \beta) \leq 2H(\alpha)H(\beta). \tag{13}$$

Equivalently,

$$h(\alpha\beta) \leq h(\alpha) + h(\beta). \tag{14}$$

$$h(\alpha + \beta) \leq \log 2 + h(\alpha) + h(\beta). \tag{15}$$

There is a natural notion of height for vectors of algebraic numbers which is defined in chapter 3 of [Sch99]. For our purposes it will be enough to know that if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is a vector of algebraic numbers then for all i, $1 \leq i \leq k$

$$H(\alpha_i) \leq H(\boldsymbol{\alpha}), \tag{16}$$

and

$$h(\alpha_i) \leq h(\boldsymbol{\alpha}). \tag{17}$$

A highly nontrivial result due to Showu Zhang [Zha] is as follows:

Lemma 5. *Let $P_1, P_2, \cdots, P_n$ be polynomials in the variables $X_1, X_2, \cdots, X_n$. Let $d_i = \deg(P_i)$. If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is an isolated solution to the equations $P_i = 0$ then its absolute logarithmic height is bounded as follows:*

$$h(\boldsymbol{\alpha}) \leq A(n) \left(\sum_{i=1}^{n} d_i \right) \left(\left(\sum_{i=1}^{n} \frac{M(P_i)}{d_i} \right) + B(n) \log 2 \right)$$

where $A(n) = (n^2 - n + 1)! n^{-2} [(n-1)!]^{-n}$ and $B(n) = (n^3 - n^2)(n^2 - n + 1)^{-1}$.

Finally, a technical result that will be used in the discussion of the hyperbolic pieces is as follows:

Lemma 6. *Suppose α_{11}, α_{12}, and α_{22} are algebraic numbers such that the upper triangular 2×2 matrix,*

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix},$$

is the product, $B_1 B_2 \cdots B_k$, of the k matrices,

$$B_i = \begin{pmatrix} \beta_{11}^i & \beta_{12}^i \\ 0 & \beta_{22}^i \end{pmatrix},$$

with each β_{jl}^i an algebraic number. Suppose further that $h(\beta_{jl}^i) \leq h$ for all i, j, l. Then

$$h(\alpha_{11}), h(\alpha_{22}) \leq kh,$$

and

$$h(\alpha_{12}) \leq (k-1)\log 2 + \left(\frac{1}{2}k^2 + \frac{3}{2}k - 1\right) h.$$

Proof. The proof will proceed by induction on k. Clearly the lemma holds for the case $k = 1$. Now suppose it is true for $A' = B_1 B_2 \cdots B_{k-1}$. Let $A = B_1 B_2 \cdots B_{k-1} B_k = A' B_k$. Notice that

$$\begin{aligned} A &= \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix} \\ &= \begin{pmatrix} \alpha'_{11} & \alpha'_{12} \\ 0 & \alpha'_{22} \end{pmatrix} \begin{pmatrix} \beta^k_{11} & \beta^k_{12} \\ 0 & \beta^k_{22} \end{pmatrix} \\ &= \begin{pmatrix} \alpha'_{11}\beta^k_{11} & \alpha'_{11}\beta^k_{12} + \alpha'_{12}\beta^k_{22} \\ 0 & \alpha'_{22}\beta^k_{22} \end{pmatrix}. \end{aligned} \tag{18}$$

Consequently,

$$\begin{aligned} h(\alpha_{11}) &= h(\alpha'_{11}\beta^k_{11}) \\ &\leq h(\alpha'_{11}) + h(\beta^k_{11}) \\ &\leq (k-1)h + h \\ &= kh. \end{aligned} \tag{19}$$

Similarly, $h(\alpha_{22}) \leq kh$. Now consider $h(\alpha_{12})$.

$$\begin{aligned} h(\alpha_{12}) &= h(\alpha'_{11}\beta^k_{12} + \alpha'_{12}\beta^k_{22}) \\ &\leq \log 2 + h(\alpha'_{11}\beta^k_{12}) + h(\alpha'_{12}\beta^k_{11}) \\ &\leq \log 2 + h(\alpha'_{11}) + h(\beta^k_{12}) + h(\alpha'_{12}) + h(\beta^k_{22}) \\ &\leq \log 2 + (k-1)h + h + (k-2)\log 2 \\ &\quad + \left(\frac{1}{2}(k-1)^2 + \frac{3}{2}(k-1) - 1\right) h + h \qquad (20) \\ &= (k-1)\log 2 + (k+1)h + \left(\frac{1}{2}k^2 + \frac{1}{2}k - 2\right) h \\ &= (k-1)\log 2 + \left(\frac{1}{2}k^2 + \frac{3}{2}k - 1\right) h. \end{aligned}$$

□

3.2 Covering hyperbolic pieces

We are now ready to produce the desired covers of the hyperbolic pieces in the JSJ decomposition of our knot complement. Let $D(n)$ be as in (3).

Theorem 2. *Suppose $N - \partial N$ has a complete, finite volume hyperbolic structure and ∂N has $b \in \{1, 2\}$ components. If N has a combinatorial ideal triangulation with n tetrahedra then for every prime $p > D(n)$, N has a finite cover $\widetilde{N}$ with at most $p^{\left(108n^2 24^n + 2\right)}$ sheets in which each boundary component of $\widetilde{N}$ is the noncyclic cover of order p^2 of a boundary component of N.*

Proof. Suppose $N - \partial N$ has a complete, finite volume hyperbolic structure and ∂N has $b \in \{1, 2\}$ components. Let us also suppose N has a combinatorial ideal triangulation, $\mathcal{T} = \bigcup_{i=1}^{n} \sigma_i$, with n ideal tetrahedra.

A hyperbolic ideal tetrahedron is specified up to isometry by a single complex parameter. We will use the convention that complex parameters in the upper half-plane specify positively oriented ideal hyperbolic tetrahedra, parameters on the

real line which are not equal to 0 or 1 represent "flat" ideal hyperbolic tetrahedra, and parameters in the lower half-plane represent negatively oriented ideal hyperbolic tetrahedra. At times I will refer to "degenerate" hyperbolic tetrahedra. This will mean that two or more of the vertices on the Riemann sphere at infinity are identified. Note that flat tetrahedra will not be considered degenerate.

By [EP88] $N - \partial N$ can be triangulated by positively oriented and flat ideal hyperbolic tetrahedra realizing the hyperbolic structure of N. Let $\mathcal{T}_0$ denote this triangulation. From [Pie88] it follows that we can relate $\mathcal{T}$ and $\mathcal{T}_0$ by 2-3 and 3-2 bi-stellar moves. Fix some such sequence of moves and use it to assign (possibly degenerate) hyperbolic structures to each of the ideal tetrahedra in $\mathcal{T}$. A degenerate tetrahedron will be one for which two or more of the vertices at infinity are identified. Create a new ideal triangulation $\mathcal{T}'$ (of what at first may appear to be a new 3-manifold N') as follows: If σ_i has a degenerate structure, identify the points of σ_i in $\mathcal{T}$ which are identified in this structure. Then $\mathcal{T}'$ will be an ideal triangulation with n or less ideal tetrahedra.

Assuming this has been done replace $\mathcal{T}$ with $\mathcal{T}'$. We are now assured that $\mathcal{T}$ has no degenerate hyperbolic ideal tetrahedra. Next we examine certain polynomial equations called the gluing and completeness equations [NZ85]. There is one gluing equation for each edge of the triangulation of N, and it asserts that the product of the parameters of the hyperbolic ideal tetrahedra about the edge is 1. There is one completeness equation for each boundary component of N and it asserts that the product of the parameters along an essential loop in a boundary component is 1. Our 3-manifold N has torus boundary and consequently its Euler characteristic must be 0. This shows that our ideal triangulation with n tetrahedra must have n edges after identification. Let $G_1(X_1, \cdots, X_n) = 0$, $\cdots$, $G_n(X_1, \cdots, X_n) = 0$ be the gluing equations and $C_1(X_1, \cdots, X_n) = 0$, $\cdots$, $C_b(X_1, \cdots, X_n) = 0$ be the completeness equations for $\mathcal{T}$. There is a common root, $\mathbf{z} = (z_1, z_2, \cdots, z_n)$, of these

equations that corresponds to the hyperbolic structure N. Mostow rigidity implies that $\mathbf{z}$ is, in fact, an isolated solution. According to Proposition 2.3 of [NZ85], b of the gluing equations are redundant and so after possibly renumbering the gluing equations, $\mathbf{z}$ is an isolated solution to the n simultaneous equations, $G_1(X_1, \cdots, X_n) = 0, \cdots, G_{n-b}(X_1, \cdots, X_n) = 0$ and $C_1(X_1, \cdots, X_n) = 0, \cdots, C_b(X_1, \cdots, X_n) = 0$.

We can construct a representation $\rho : \pi_1(N) \to SL_2(\mathbb{C})$ as follows: Consider the universal cover, $\tilde{N}$, of N. There is an equivariant triangulation, $\tilde{\mathcal{T}}$, of $\tilde{N}$ which is a lift of the triangulation $\mathcal{T}$. Fix some tetrahedron $\tilde{\sigma}_1$ of $\tilde{\mathcal{T}}$ covering σ_1. Let $d : \tilde{N} \to \mathbb{H}^3$ be the continuous map such that $d(\tilde{\sigma}_1)$ is the ideal tetrahedron with vertices 0, 1, ∞, and z_1 and for any other ideal tetrahedron $\tilde{\sigma}_k$ of $\tilde{\mathcal{T}}$ covering σ_k, $d(\tilde{\sigma}_k)$ is an ideal tetrahedron with parameter z_k. Note that d will be a bijection if and only if all the z_i are in the upper half-plane or all are in the lower half-plane. Fix a base point x for N in the interior of σ_1 and let $\tilde{x}_0$ be the lift of x in $\tilde{\sigma_1}$. For a loop $\alpha \in \pi_1(N, x)$, let $\tilde{\alpha}$ be the lift of α starting at $\tilde{x}_0$. Let $\tilde{x}_1$ be the terminal point of $\tilde{\alpha}$. The point, $\tilde{x}_1$, is in some tetrahedron $\tilde{\sigma}'_1$ of $\tilde{\mathcal{T}}$. Let $\rho'(\alpha)$ be the hyperbolic isometry taking $d(\tilde{\sigma}_1)$ to $d(\tilde{\sigma}'_1)$. Notice that this representation $\rho' : \pi_1(N) \to PSL_2(\mathbb{C})$ agrees with the representation arising in the same manner from the original triangulation $\mathcal{T}_0$. Thus ρ' is the representation of $\pi_1(N)$ coming from the hyperbolic structure of N. This shows that collapsing degenerate tetrahedra did not actually change the topological type of N.

For each z_k consider the orientation preserving hyperbolic isometries which take a face of an ideal tetrahedron with vertices 0, 1, ∞, and z_k and moves it so that it coincides with another face of this tetrahedron. A quick calculation gives that such isometries will be compositions of the linear fractional transformations corresponding to the following matrices and their inverses:

$$\begin{pmatrix} \sqrt{z_k} & 0 \\ 0 & \frac{1}{\sqrt{z_k}} \end{pmatrix} \quad \text{Takes } (0, 1, \infty) \text{ to } (0, z_k, \infty) \tag{21}$$

$$\begin{pmatrix} \sqrt{z_k - 1} & \frac{1}{\sqrt{z_k - 1}} \\ 0 & \frac{1}{\sqrt{z_k - 1}} \end{pmatrix} \quad \text{Takes } (0, 1, \infty) \text{ to } (1, z_k, \infty) \tag{22}$$

$$\begin{pmatrix} \sqrt{\frac{z_k}{1-z_k}} & 0 \\ \sqrt{\frac{z_k}{1-z_k}} & \sqrt{\frac{1-z_k}{z_k}} \end{pmatrix} \quad \text{Takes } (0, 1, \infty) \text{ to } (0, z_k, 1) \tag{23}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{Takes } (0, 1, \infty) \text{ to } (1, \infty, 0) \tag{24}$$

$$\begin{pmatrix} -i & i \\ 0 & i \end{pmatrix} \quad \text{Takes } (0, 1, \infty) \text{ to } (1, 0, \infty) \tag{25}$$

The choices of the roots $\sqrt{z_k}$ and $\sqrt{z_k + 1}$ can be made arbitrarily and will determine how the representation, $\rho' : \pi_1(N) \to PSL_2(\mathbb{C})$, will be lifted to a representation, $\rho : \pi_1(N) \to SL_2(\mathbb{C})$.

I claim that $\rho(\pi_1(N))$ will be contained in the subgroup of $SL_2(\mathbb{C})$ generated by the above matrices and their inverses. The above linear fractional transformations allow one to take any face of an ideal tetrahedron with vertices 0, 1, ∞ and z_k to the ideal triangle with vertices 0, 1, ∞. Thus a composition of the above transformations will take any face in the "tessellation", $\tilde{\mathcal{T}}$, of $\mathbb{H}^3$ to any other. In particular all face identifications of some fundamental domain will be products of the above matrices and their inverses.

Let A be the subring of $\mathbb{C}$ generated by all entries of the above matrices for $k = 1, \cdots, n$. Then since inversion in SL_2 can be performed without division, we can say that $\rho(\pi_1(N)) \subset SL_2(A)$. For each boundary component of N let $\mu_j, \lambda_j \in \pi_1(N)$ be a meridian and longitude respectively with the convention that the longitude be trivial in the homology of N. By possibly modifying our choice of σ_1 we may assume

that there are nonzero $m_1, l_1 \in A$ such that

$$\rho(\mu_1) = \begin{pmatrix} 1 & m_1 \\ 0 & 1 \end{pmatrix} \tag{26}$$

and

$$\rho(\lambda_1) = \begin{pmatrix} 1 & l_1 \\ 0 & 1 \end{pmatrix}. \tag{27}$$

Also for $j > 1$ there are $P_j \in SL_2(A)$ and nonzero $m_j, l_j \in A$ such that

$$P_j \rho(\mu_j) P_j^{-1} = \begin{pmatrix} 1 & m_j \\ 0 & 1 \end{pmatrix} \tag{28}$$

and

$$P_j \rho(\lambda_j) P_j^{-1} = \begin{pmatrix} 1 & l_j \\ 0 & 1 \end{pmatrix}. \tag{29}$$

If we can produce a ring homomorphism, $\eta : A \to F$, for some finite field, F, of characteristic p for which $\eta(l_j) \neq 0$ for each j then this will induce a group homomorphism, $\eta^* : SL_2(A) \to SL_2(F)$, with the property that $\eta^*(\rho(\lambda_j))$ has order exactly p. We can then take the product of $\eta^* \circ \rho$ with the map $\theta : \pi_1(N) \to H_1(M)/pH_1(M)$ which is the Hurewicz homomorphism modulo p. The kernel of $(\eta^* \circ \rho) \times \theta$ restricted to fundamental group of boundary component j will be generated by λ_j^p and μ_j^p. Hence the cover of N corresponding to the kernel of $(\eta^* \circ \rho) \times \theta$ will have the desired boundary components.

To this end let us consider what conditions on p ensure that such a homomorphism, η, exists. Let $R_k(X) \in \mathbb{Z}[X]$ be the minimal polynomial of z_k. Note that the sum of the coefficients of R_k cannot be 0 or else $X - 1$ divides R_k ($R_k \neq X - 1$ since $z_k \neq 1$). Fix a prime p and let $R_k^{(p)}(X)$ denote the image of $R_k(X)$ under the natural map $\mathbb{Z}[X] \to \mathbb{F}_p[X]$ where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. If p does not divide the leading coefficient of $R_1(X)$ then $R_1^{(p)}(X)$ has a root ζ_1 in the algebraic closure of $\mathbb{F}_p$, and we have a ring homomorphism $\mathbb{Z}[z_1] \to \mathbb{F}_p(\zeta_1)$ taking z_1 to ζ_1. If, in addition, p does not divide the

constant term of $R_1(X)$ then we are assured that ζ_1 is nonzero and is hence invertible. Finally if p does not divide the sum of the coefficients of R_1 then $\zeta_1 - 1$ cannot be 0 and must be invertible. Set $A_1 = \mathbb{Z}[z_1, 1/z_1, 1/(z_1 - 1)]$ and $F_1 = \mathbb{F}_p(\zeta_1)$ In this case we get a ring homomorphism, $\eta_1 : A_1 \to F_1$. Let $\tilde{\eta}_1 : A_1[X] \to F_1[X]$ be the induced map on the polynomial rings.

Inductively let $Q_{i+1}(X) \in A_i[X]$ be the minimal polynomial of z_{i+1} over A_i. If p does not divide the leading coefficient of $R_{i+1}(X)$ then $R_{i+1}^{(p)}(X)$ has roots in the algebraic closure of $\mathbb{F}_p$ some of which will be roots of $\tilde{\eta}_i\left(Q_{i+1}(X)\right)$. Let ζ_{i+1} be one such root. If, in addition, p does not divide the constant term of $R_{i+1}(X)$ then we are assured that ζ_{i+1} is nonzero and hence invertible. As before, if p does not divide the sum of the coefficients of R_{i+1} then $\zeta_{i+1} - 1$ cannot be 0 and must be invertible. Set $A_{i+1} = A_i[z_{i+1}, 1/z_{i+1}, 1/(z_{i+1} - 1)]$ and $F_{i+1} = F_i(\zeta_{i+1})$. In this case we get a ring homomorphism $\eta_{i+1} : A_{i+1} \to F_{i+1}$. Let $\tilde{\eta}_{i+1} : A_{i+1}[X] \to F_{i+1}[X]$ be the induced map on the polynomial rings.

From this discussion it is clear that we will have a homomorphism, $\eta_n : A_n \to F_n$, if p does not divide any of the constant coefficients of the R_k's, the leading coefficients of the R_k's, or the sums of the coefficients of the R_k's.

A contains the roots $\sqrt{z_k}$ and $\sqrt{z_k - 1}$ so we require F to contain $\sqrt{\zeta_k}$ and $\sqrt{\zeta_k - 1}$. Start with F_n and inductively adjoin appropriate roots of the equations $X^2 - \zeta_k = 0$ and $X^2 - (\zeta_k - 1) = 0$ whenever necessary. This can be done without any further restrictions on p. One quickly arrives at the desired homomorphism, $\eta : A \to F$.

We will now proceed to bound the degree $[F : \mathbb{F}_p]$. Let Ω_p be the algebraic closure of $\mathbb{F}_p$ and $\varphi : \Omega_p \to \Omega_p$ be the Frobenius mapping given by $\varphi(x) = x^p$. Note that φ generates the group of automorphisms of Ω_p over $\mathbb{F}_p$. If $\varphi^m(x) = x$ then x is in $\mathbb{F}_{p^m} \subset \Omega_p$ (See [Lan93] pg. 246). Consider the orbit of $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \cdots, \zeta_n)$ under

the map

$$\hat{\varphi}(\zeta_1, \zeta_2, \cdots, \zeta_n) = (\varphi(\zeta_1), \varphi(\zeta_2), \cdots, \varphi(\zeta_n)). \tag{30}$$

$\boldsymbol{\zeta}$ is an isolated solution of the gluing and completeness equations modulo p and hence every point $\hat{\varphi}^k(\boldsymbol{\zeta})$ must be as well. The product of the degrees of the gluing and completeness equations bounds the number of isolated solutions; hence, the orbit of $\boldsymbol{\zeta}$ has order at most the product of these degrees. This shows that for some m less than or equal to the product of the degrees of the gluing and completeness equations $\hat{\varphi}^m(\boldsymbol{\zeta}) = \boldsymbol{\zeta}$. In other words $\varphi^m(\zeta_i) = \zeta_i$ for all i. It follows that $\zeta_1, \zeta_2, \cdots, \zeta_n \in \mathbb{F}_{p^m}$ and hence $F_n \subset \mathbb{F}_{p^m}$. Consequently,

$$[F_n : \mathbb{F}_p] \leq \prod_{i=1}^{n-b} \deg G_i \prod_{i=1}^{b} \deg C_i. \tag{31}$$

We get F from F_n by adjoining at most $2n$ square roots so

$$[F_n : F] \leq 2^{2n}. \tag{32}$$

We can combine the above inequalities to get that

$$[F : \mathbb{F}_p] \leq 2^{2n} \prod_{i=1}^{n-b} \deg G_i \prod_{i=1}^{b} \deg C_i. \tag{33}$$

Each polynomial G_i will have degree less than or equal to the order of the edge it comes from, and each of the six edges of a tetrahedron appears once in a gluing equation. Whence,

$$\sum_{i=1}^{n} \deg G_i \leq 6n. \tag{34}$$

For each boundary component the triangulation $\mathcal{T}$ gives a corresponding triangulation of the boundary torus. Choose a point in the boundary in the interior of one of these triangles and pick a nontrivial loop passing through the fewest triangles. Such a loop passes through a triangle at most once. For the ith boundary component, C_i is the completeness equation that asserts that product of all the edge parameters "to the right" of the loop is 1 (See [NZ85] for a detailed description of the completeness

equations). Each tetrahedron has 4 corners so the total number of corners of triangles in the triangulations of the boundary tori is $3 \cdot 4n$. This shows that

$$\sum_{i=1}^{b} \deg C_i \leq 12n. \tag{35}$$

The product of k positive integers whose sum is fixed is greatest if all the numbers are equal; hence, (34), (35), and $b \leq 2$ imply that

$$\prod_{i=1}^{n} \deg G_i \leq 6^n \tag{36}$$

and

$$\prod_{i=1}^{b} \deg C_i \leq 36n^2. \tag{37}$$

The above two inequalities combine with (33) to show that

$$[F : \mathbb{F}_p] \leq 2^{2n} 6^n 36n^2 = 36n^2 24^n. \tag{38}$$

We also get a bound of the order of the field F.

$$|F| \leq p^{\left(36n^2 24^n\right)}. \tag{39}$$

The order of $SL_2(F)$ is $(|F|^2 - 1)(|F| - 1)$; hence,

$$|SL_2(F)| \leq p^{\left(108n^2 24^n\right)}. \tag{40}$$

We now address the question of how to ensure that p meets all of the conditions stipulated above. This will be done using the Malher measures of the gluing and completeness equations. To get a gluing equation we take the product around an edge of X_k's, $\frac{1}{1-X_k}$'s, and $\frac{X_k - 1}{X_k}$'s and set it equal to 1. Clearing denominators the equation becomes:

$$J_1 J_2 \cdots J_{\deg G_i} = L_1 L_2 \cdots L_{\deg G_i} \tag{41}$$

Where $J_i = X_{k_i}$, $(X_{k_i} - 1)$, or 1 and $L_i = (1 - X_{k_i})$, X_{k_i}, or 1. Thus each G_i is of the

form $J_1 J_2 \cdots J_{\deg G_i} - L_1 L_2 \cdots L_{\deg G_i}$.

$$\begin{aligned}
\|G_i\| &= \|J_1 J_2 \cdots J_{\deg G_i} - L_1 L_2 \cdots L_{\deg G_i}\| \\
&\leq \|J_1 J_2 \cdots J_{\deg G_i}\| + \|L_1 L_2 \cdots L_{\deg G_i}\| \\
&\leq \|J_1\| \cdot \|J_2\| \cdot \cdots \cdot \|J_{\deg G_i}\| + \|L_1\| \cdot \|L_2\| \cdot \cdots \cdot \|L_{\deg G_i}\| \\
&\leq \sqrt{2}^{\deg G_i} + \sqrt{2}^{\deg G_i} \qquad (42) \\
&\leq 2\sqrt{2}^{6n} \\
&= 2^{3n+1}
\end{aligned}$$

Completeness equations arise in the same manner except that they assert that the product of the X_k's, $\frac{1}{1-X_k}$'s, and $\frac{X_k - 1}{X_k}$ along some essential loop in the boundary is 1. Thus the C_i's are of the same form as the G_i's and the same argument gives that $\|C_i\| \leq 2\sqrt{2}^{\deg C_i} \leq 2^{6n+1}$. From Lemma 3 we get

$$M(G_i) \leq 2^{3n+1} \qquad (43)$$

and

$$M(C_i) \leq 2^{6n+1}. \qquad (44)$$

Recall that $\mathbf{z} = (z_1, \cdots, z_n)$ is our set of edge parameters. According to Lemma 5 if $A(n)$ and $B(n)$ are as in (1) and (2) then

$$\begin{aligned}
h(\mathbf{z}) &\leq A(n) \left(\sum_{i=1}^{n-b} \deg G_i + \sum_{i=1}^{b} \deg C_i \right) \\
&\quad \cdot \left(\left(\sum_{i=1}^{n-b} \frac{M(G_i)}{\deg G_i} + \sum_{i=1}^{b} \frac{M(C_i)}{\deg C_i} \right) + B(n) \log 2 \right) \\
&\leq A(n)\,(6n + 12n) \left(\left(\sum_{i=1}^{n} 2^{3n+1} + \sum_{i=1}^{2} 2^{6n+1} \right) + B(n) \log 2 \right) \qquad (45) \\
&\leq A(n)(18n) \left(\left(n 2^{3n+1} + 2^{6n+2} \right) + B(n) \log 2 \right).
\end{aligned}$$

For every k, $h(z_k) \leq h(\mathbf{z})$; therefore,

$$h(z_k) \leq A(n)(18n) \left(n 2^{3n+1} + 2^{6n+2} + B(n) \log 2 \right). \qquad (46)$$

In order to convert the height of z_k into its Mahler measure one needs to know its degree. In [Dub90] a bound on the degree of a polynomial in a Gröbner basis with any monomial order is given. This gives a bound on the degrees of the polynomials in a Gröbner basis of the ideal generated by the gluing and completeness equations which in turn gives a bound on the degree of z_k.

$$\deg(z_k) \leq 2\left(72n^2 + 12n\right)^{2^{n-1}}. \tag{47}$$

We can now use (11) to bound the Mahler measure of z_k.

$$M(z_k) \leq \exp\left[2\left(72n^2 + 12n\right)^{2^{n-1}} A(n)(18n)\left(n2^{3n+1} + 2^{6n+2} + B(n)\log 2\right)\right]. \tag{48}$$

Using this bound we see that if $R_k(X)$ is the minimal polynomial for z_k over $\mathbb{Z}$ then by Lemma 4, R_k must have length bounded by

$$\begin{aligned} L(R_k) \;\leq\; & \exp\Big[2\left(72n^2 + 12n\right)^{2^{n-1}} (\log 2 + A(n)(18n) \\ & \cdot \left(n2^{3n+1} + 2^{6n+2} + B(n)\log 2\right)\Big)\Big]. \end{aligned} \tag{49}$$

If p is larger than $L(R_k)$ then it cannot divide the first and last coefficients of R_k or $R_k(1)$. This is sufficient to ensure that we have a ring homomorphism $\eta : A \to F$. However, $\eta(l_i)$ may still be 0 for some i with $1 \leq i \leq b$.

In the discussion on the degrees of the completeness equations it was mentioned that we could choose generators for the fundamental groups of the boundary components which were products of at most $12n$ matrices of type (21) or (22). In the next section it will be shown that the meridian and longitude can be expressed as words of length less than 3^n in these generators. Consequently, for each i, $\rho(\lambda_i)$ is a products of at most $12n3^n$ matrices of type (21) or (22). We can use the above bound on the heights of the z_k's and Lemma 6 to bound the height of the upper right

entry, l_i, of $\rho(\lambda_i)$. We can thus conclude that

$$\begin{aligned} h(l_i) &\leq (12n3^n - 1)\log 2 + A(n)(18n)\left(n2^{3n+1} + 2^{6n+2} + B(n)\log 2\right) \\ &\quad \cdot(72n^2 3^{2n} + 18n3^n - 1). \end{aligned} \tag{50}$$

The degree of l_i is at most the product of the degrees of the z_k's; hence,

$$\deg(l_i) \leq 2\left(72n^2 + 12n\right)^{n2^{n-1}}. \tag{51}$$

From (11) we get:

$$\begin{aligned} M(l_i) &\leq \exp\Big[2\left(72n^2 + 12n\right)^{n2^{n-1}}((12n3^n - 1)\log 2 + A(n)(18n) \\ &\quad \cdot\left(n2^{3n+1} + 2^{6n+2} + B(n)\log 2\right)(72n^2 3^{2n} + 18n3^n - 1))\Big]. \end{aligned} \tag{52}$$

We wish $D(n)$ to be the maximum of the bounds in (49) and (52). The dominant bound is (52) so we choose $D(n)$ to be (52). If $p > D(n)$ then, as described above, we may produce a finite field, F, and a homomorphism, $\eta : A \to F$, such that $\eta(l_i)$ is nonzero. It follows that in the induced map $\eta^* : SL_2(A) \to SL_2(F)$, the order of $\eta^*(\rho(\lambda_i))$ is p. We cross the map $\eta^* \circ \rho$ with the map $\theta : \pi_1(N) \to H_1(N) \to H_1(N)/pH_1(N)$. The kernel of the map $(\eta^* \circ \rho) \times \theta : \pi_1(N) \to SL_2(F) \times (\mathbb{Z}/p\mathbb{Z})^b$ corresponds to a regular cover, $\widetilde{N}$, of N whose boundary components are the noncyclic covers of index p^2 of the boundary components of N. In light of (40) we get a bound on the order of $SL_2(F) \times (\mathbb{Z}/p\mathbb{Z})^b$.

$$\left|SL_2(F) \times (\mathbb{Z}/p\mathbb{Z})^b\right| \leq p^{\left(108n^2 24^n + 2\right)}. \tag{53}$$

This is also bound on the number of sheets of $\widetilde{N}$. □

4 Seifert Fibered Pieces

4.1 Seifert fibered pieces of a knot complement

We now turn our attention to Seifert fibered pieces of our knot complement.

Lemma 7. *Suppose N is a piece in the JSJ decomposition of the complement of a nontrivial knot in S^3, and N is Seifert fibered. Then N has base orbifold either a disk with two cone points of order u and v with $(u, v) = 1$, or N has base orbifold an annulus with one cone point.*

Proof. As in the statement of the lemma we assume N is a Seifert fibered piece in the JSJ decomposition of the complement of a knot in S^3. Then N has one or two boundary components.

Case 1. Assume N has one boundary component. Then since N is embedded in S^3, the boundary of N is a torus in S^3. By the Solid Torus Theorem (see [Rol90] pg. 107), $\partial N \subset S^3$ bounds a solid torus on at least one side. Clearly, N cannot be a solid torus (since we assumed the knot to be nontrivial), so $S^3 - N$ must be a solid torus. This shows that N is actually the complement of a knot, K', in S^3. The only knots with Seifert fibered complements are torus knots (See Theorem 10.5.1 of [Kaw96]). Hence, for some relatively prime $u, v \in \mathbb{Z}$, the knot K' is the (u, v)-torus knot. It follows that N has a Seifert fibered structure with two singular fibers with orders u and v. Thus, the base orbifold of N is a disk with two cone points with relatively prime orders u and v.

Case 2. Assume N has two boundary components. The manifold, N is *simple* in the sense that it contains no essential tori (otherwise we would have cut along them). As in Proposition C.5.2 of [Kaw96], the only simple Seifert fibered manifolds with two boundary components have base orbifold an annulus with a single cone point. □

4.2 Seifert fibered pieces with two boundary components

We will now produce bounds for the orders of the cone points in the above lemma.

Lemma 8. *Suppose N is a piece in the JSJ decomposition of the complement of a nontrivial knot, K, in S^3, and N is a Seifert fibered space with base orbifold an annulus with a cone point of order u. If K has a diagram with c crossings then $u \leq \frac{c}{2}$.*

Proof. Let K be a nontrivial knot in S^3 and N be a Seifert fibered piece of the JSJ decomposition of $M = S^3 - K$. Further, suppose that the base orbifold of N is an annulus with a cone point of order u, so $N \subset M$ has two boundary components. Let S be the boundary component of N which is closer to ∂M. Let M' be the component of $M - S$ which does not intersect ∂M. M' is the complement of a knot K' in S^3, and K is a satellite of K'. By Theorem 3 of [Sch54] the bridge number, b', of K' must be less than or equal to the bridge number, b, of K.

Consider the knot K'. It has a satellite knot decomposition with pattern a (u, q)-torus knot in a standard solid torus for some q. Let K'' be the corresponding companion, and let b'' be its bridge number. The winding number of the (u, q)-torus knot in the solid torus is u. Using Theorem 3 of [Sch54] we can conclude that

$$ub'' \leq b'. \tag{54}$$

Clearly $2 \leq b''$ so

$$2u \leq ub'' \leq b' \leq b. \tag{55}$$

The bridge number of a knot must be less than or equal to the crossing number of any projection of the knot so $b \leq c$; hence, we get the desired result:

$$2u \leq c \tag{56}$$

□

4.3 Seifert fibered pieces with one boundary component

Now one case remains. If our knot K is a satellite of the (u, v)-torus knot then the JSJ decomposition of its complement has a Seifert fibered piece whose base space is a disk with two cone points. We will bound u and v based on the crossing number of K. The bridge number of K must be greater than the bridge number of the (u, v)-torus knot. The bridge number of a torus knot is known to be the smaller of $|u|$ and $|v|$ (See Theorem 7.5.3 of [Mur96]); consequently, if c is the number of crossings in some diagram of K then the smaller of $|u|$ and $|v|$ must be less than c. Unfortunately, we must bound the larger of the two.

Recall that Lemma 2 gives a bound on the number of tetrahedra needed to triangulate any piece in the JSJ decomposition of the complement of the knot K. We will proceed to bound the minimum number of tetrahedra needed to triangulate the complement of a (u, v)-torus knot from below. This will be done by showing that a knot complement that can be triangulated with n ideal tetrahedra has an Alexander polynomial with degree at most $(n^2 + n)3^{n+1}$.

Lemma 9. *Suppose K is a knot in S^3, and its complement $M = S^3 - K$ can be triangulated with n ideal tetrahedra. Then the Alexander polynomial of K has degree at most $(n^2 + n)3^{n+1}$.*

Proof. Let K be a knot in S^3 and $N = S^3 - K$ its complement. Suppose N has an ideal triangulation $\mathcal{T}$ with n ideal tetrahedra. Set $G = \pi_1(N)$.

N has a standard spine C which is dual to $\mathcal{T}$. C will have n vertices and $4n/2 = 2n$ edges. Note that C has the same homology as S^1 so clearly the Euler characteristic of C is 0. This implies that C must have n faces. If we fix a maximal tree in the 1-skeleton of C, we get a presentation $\langle S_0, S_1, \ldots, S_n | R_1, R_2, \ldots, R_n \rangle$ for G with $n + 1$ generators and n relations. Furthermore, each edge of C is incident with 3 faces so the sum of the lengths of the relations must be $3(n + 1)$.

Following the technique given in [BZ85] Example 9.15 this presentation of the group may be used to find the first elementary ideal of the Alexander module of K. To begin we let $F = \langle S_0, S_1, \ldots, S_n \rangle$ be the free group. Let the derivations $\frac{\partial}{\partial S_i} : \mathbb{Z}F \to \mathbb{Z}F$ be the linear maps satisfying the following rules for all $\alpha, \beta \in F$:

1. $\frac{\partial}{\partial S_i}(S_j) = \delta_{ij}$.
2. $\frac{\partial}{\partial S_i}(\alpha^{-1}) = \alpha^{-1}\frac{\partial}{\partial S_i}(\alpha)$.
3. $\frac{\partial}{\partial S_i}(\alpha \cdot \beta) = \frac{\partial}{\partial S_i}(\alpha) + \alpha\frac{\partial}{\partial S_i}(\beta)$.

For each generator S_i and each relation R_j compute $\frac{\partial}{\partial S_i}R_j$. Let $\psi : \mathbb{Z}F \to \mathbb{Z}G$ be the linear extension of the quotient homomorphism, $F \to G$, and $\varphi : \mathbb{Z}G \to \mathbb{Z}\langle t \rangle$ be the linear extension of the Hurewicz homomorphism, $G \to H_1(N) = \langle t \rangle$. By [BZ85] Proposition 9.14 we know that the ideal of $\mathbb{Z}\langle t \rangle$ generated by the determinants of the $n \times n$ minors of $A = \left(\varphi \circ \psi(\frac{\partial R_j}{\partial S_i})\right)$ will be a principal ideal generated by the Alexander polynomial, $\Delta(t)$, of K. Let A_i be the $n \times n$ minor of A got by removing the ith column. $\Delta(t)$ divides $\det A_i$ so if $d = \max\{\deg(\det A_i)\}$ then $\deg \Delta(t) \leq d$. If each entry of A has degree l or less then $\deg(\det A_i) \leq nl$, so all that remains is to bound the degrees of the entries of A.

Let us consider an entry, $a_{ij} = \varphi \circ \psi(\frac{\partial R_j}{\partial S_i})$, of A. As noted above, R_j is a word in the S_k's of length at most $3(n+1)$. Applying the rules above it is clear that $\frac{\partial}{\partial S_i}R_j$ is a linear combination of words in the S_k's with lengths bounded by $3(n+1)$. The map $\varphi \circ \psi$ takes a word in the S_k's to t^a where $a \in \mathbb{Z}$ is the number of times the path represented by the word winds around the knot K. Choose the integers ν_k so that $\varphi \circ \psi(S_k) = t^{\nu_k}$. If $|\nu_k| \leq \nu$ for all k then $\deg(a_{ij}) \leq \nu 3(n+1)$.

Let us now proceed to produce such a number ν. Each relation, R_j, is trivial in G and hence must map to the identity in $H_1(N)$. In other words, $\varphi \circ \psi(R_j) = t^0$. This translates to an equation specifying that some integral linear combination of ν_k's is 0. For example, the relation $S_0S_2S_3S_1^{-1}S_2 = 1$ would give the equation

$1\nu_0 - 1\nu_1 + 2\nu_2 + 1\nu_3 = 0$. Consider the $n \times (n+1)$ matrix, B, whose ijth entry is the coefficient of ν_j in the equation coming from the relation R_i. The vector $(\nu_0, \cdots, \nu_n)$ will be the smallest nonzero, integral vector whose dot product with each row of B is 0. Since B is a presentation matrix for the homology of N, this property actually characterizes $(\nu_0, \cdots, \nu_n)$ up to sign. Let B_j be the $n \times n$ minor of B formed by dropping the jth column of B. I claim that the integer vector $(\det B_0, -\det B_1, \det B_2, \cdots, (-1)^n \det B_n)$ is a multiple of $(\nu_0, \cdots, \nu_n)$. Let $\mathbf{x} = (x_0, \cdots, x_n)$ be an arbitrary vector. By definition, $\mathbf{x}$ will be in the row space of B if and only if the determinant of the $n \times n$ matrix formed by adding $\mathbf{x}$ in as the first row of B is 0. Hence, $\mathbf{x}$ will be in the row space of B if and only if $\sum_{j=0}^{n} x_j (-1)^j \det(B_j) = 0$. This shows that $(\det B_0, -\det B_1, \det B_2, \cdots, (-1)^n \det B_n)$ is indeed perpendicular to the row space of B and must be a multiple of $(\nu_0, \cdots, \nu_n)$.

Recall that in a standard spine, an edge in the 1-skeleton is incident with exactly 3 faces. This implies that the sum of the absolute values of entries in a column of B is 3. It follows that $|\det(B_j)| \leq 3^n$ for each minor B_j. We have also shown that $|\nu_j| \leq |\det(B_j)|$ for all j, thus $|\nu_j| \leq 3^n$ for all j. As noted above we get that

$$\deg(a_{ij}) \leq 3^n \cdot 3(n+1), \tag{57}$$

and hence

$$\deg \Delta(t) \leq 3^{n+1} n(n+1). \tag{58}$$

□

4.4 Covering Seifert fibered pieces

In Lemma 2 we saw that a piece in the JSJ decomposition of our manifold has at most $25 \cdot 4c = 100c$ tetrahedra. The above lemma tells us that if the piece is a knot complement then its Alexander polynomial has width at most $((100c)^2 + 100c)3^{100c+1}$.

The Alexander polynomial of a (u, v)-torus knot has degree $(u-1)(v-1)$ (See example 9.15 of [BZ85]). Clearly $u, v \geq 2$, whence, $(u-1) \leq (u-1)(v-1)$ and $(v-1) \leq (u-1)(v-1)$. From these inequalities we get that $uv = (u-1)(v-1) + (u-1) + (v-1) + 1 \leq 3(u-1)(v-1) + 1$. It follows that

$$uv \leq ((100c)^2 + 100c)3^{100c+2} + 1 \tag{59}$$

Now that we have bounded the orders of the cone points of in the base orbifolds of our Seifert fibered pieces we may proceed to produce the desired covers of these pieces.

Lemma 10. *Let N be Seifert fibered with base orbifold either a disk with two cone points of order u and v or else an annulus with one cone point of order u. Then for each prime $p > 2$, N has a cover, $\widetilde{N}$, with at most $2uvp^2$, resp. $2up^2$ sheets in which each boundary component of $\widetilde{N}$ is the noncyclic cover with p^2 sheets of a boundary component of N.*

Proof. Let F be the base orbifold of N. F is either a disk with two cone points of orders u and v or an annulus with one cone point of order u.

If F is a disk with two cone points then glue a disk with one cone point of order p to F to get a sphere F' with 3 cone points with orders u, v, and p. The orbifold, F', is a hyperbolic, and by [EEK82] has a finite orbifold cover which is a manifold. In fact, [EEK82] gives such a cover, $\widetilde{F'}$, with at most $2 \cdot \mathrm{LCM}(u, v, p)$ sheets. By removing open disk neighborhoods of each of the points of $\widetilde{F'}$ mapping to the cone point of F' with order p we get a cover $\widetilde{F}$ of F. By construction each boundary component of $\widetilde{F}$ is the p-fold cover of the boundary component of F. This shows that N has a cover, N_0, with at most $2uvp$ sheets whose base orbifold is the manifold, $\widetilde{F}$, and whose S^1 fibers map homeomorphically to the regular S^1 fibers of N. We may then take $\widetilde{N}$ to be the p-fold cover of N_0 whose base space is again $\widetilde{F}$ and whose S^1

fibers are p-fold covers of the S^1 fibers of N_0. Clearly $\widetilde{N}$ is the desired cover and has at most $2uvp^2$ sheets.

If F is an annulus with one cone point of order u, we glue two disks with cone points of order p, and by the above process, arrive at a cover $\widetilde{N}$ with at most $2up^2$ sheets. □

We now have all the ingredients needed to complete our discussion of Seifert fibered pieces.

Theorem 3. *Let N be a Seifert fibered piece in the JSJ decomposition of the complement of a nontrivial knot, K, with a diagram with c crossings. Let p be any prime greater than 3. If N has base orbifold an annulus then N has a cover $\widetilde{N}$ with cp^2 sheets or less whose boundary components are the noncyclic covers of order p^2 of the boundary components of N. Otherwise, N has orbifold a disk with two cone points. In this case, N has a cover $\widetilde{N}$ with $((20000c^2+200c)3^{100c+2}+2)p^2$ sheets or less whose boundary components are the noncyclic covers of order p^2 of the boundary components of N.*

Proof. Lemma 8 and Lemma 10 combine to give the first part of the corollary. Lemma 10 and Inequality (59) combine to give the second part. □

Notice that the bound given in Theorem 3 is exponential in the crossing number if there is a piece homeomorphic to a torus knot complement. This bound could be made polynomial if it were known that the crossing number of a satellite knot cannot be less than the crossing number of its companion. It is widely believed that this should be true, but it has remained unproven since Schubert introduced the notion of satellite knots (See Problem 1.67 of [Kir95]).

5 Assembling the Covering Space

Now that we have produced the desired covers for the geometric pieces of our knot complement, we must show that they can be assembled to produce a cover of the entire complement. This will be done exactly as in section 2 of [Hem87]. Note that this cover will in general not be regular.

Recall that K is a knot in S^3, $M = S^3 - K$ its complement, and $\{T_1, T_2, \ldots, T_r\}$ a set of tori cutting M into geometric pieces $M_0, M_1, \ldots, M_r$ (See the bottom of Figure 8). Fix a prime p. Each boundary component of M_i is a torus with fundamental group

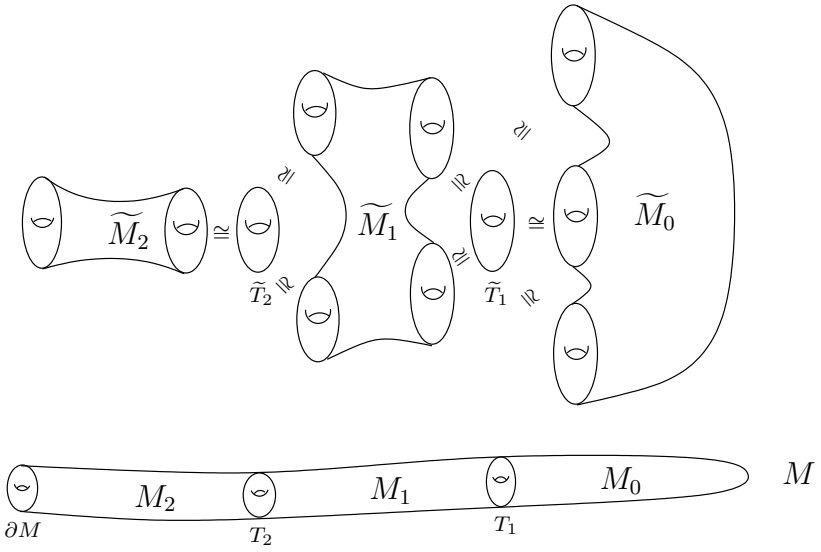


Figure 8: Pieces of the cover

isomorphic to $\mathbb{Z}\times\mathbb{Z}$. This group has a characteristic subgroup P of index p^2 generated by $(p, 0)$ and $(0, p)$. For each i, let $\widetilde{T}_i$ be the cover of T_i associated to the subgroup of $\pi_1(T_i)$ corresponding to $P \trianglelefteq \mathbb{Z} \times \mathbb{Z}$. Suppose for each i we produce a cover, $\widetilde{M}_i$, of M_i such that the boundary components of $\widetilde{M}_i$ are all covers equivalent to $\widetilde{T}_j$ for some j (see Figure 8). Then by taking sufficiently many copies, $\widetilde{M}_i^k$, of $\widetilde{M}_i$ we may assemble a cover $\widetilde{M}$ of M (see Figure 9). In fact if m_i is the number of sheets in the cover $\widetilde{M}_i$ then we will need $\mathrm{LCM}(\frac{m_0}{p^2}, \frac{m_1}{p^2}, \ldots, \frac{m_r}{p^2})/(\frac{m_i}{p^2})$ copies of $\widetilde{M}_i$, and $\widetilde{M}$ will have $p^2 \cdot \mathrm{LCM}(\frac{m_0}{p^2}, \frac{m_1}{p^2}, \ldots, \frac{m_r}{p^2})$ sheets. It is immediate that $\widetilde{M}$ is not a cyclic cover of M

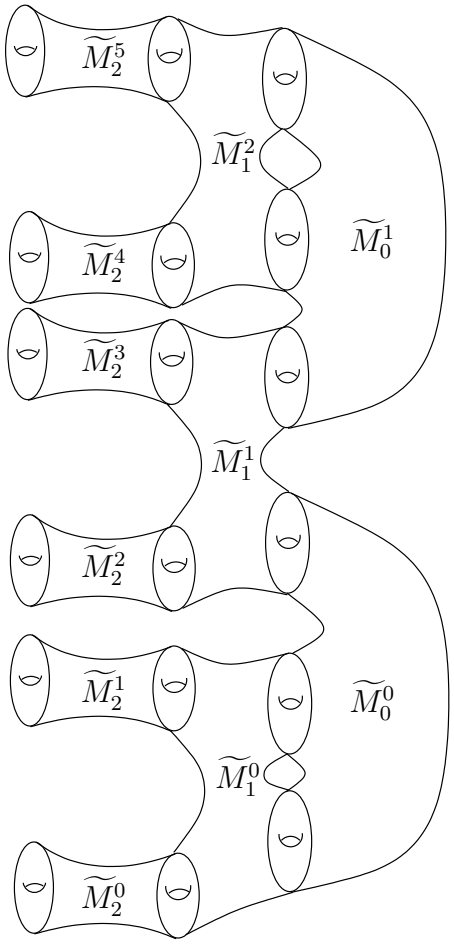


Figure 9: An assembled cover, $\widetilde{M}$, of M

since the boundary components of $\widetilde{M}$ are not cyclic covers the boundary of M.

We will now find a prime p for which we can produce such a set of coverings $\{\widetilde{M}_i\}_{i=0}^r$. Let us assume that our knot, K, has a diagram with c crossings and hence its complement, M, may be triangulated with $t \leq 4c$ tetrahedra. Corollary 1 showed that $r \leq 2t \leq 8c$ and each piece, M_i, can be triangulated with $25t \leq 100c$ or less tetrahedra. If M_i is hyperbolic then Theorem 2 implies that for every prime greater than $D(100c)$, M_i has an at most $p^{\left(1080000c^2 24^{100c}+2\right)}$-sheeted cover all of whose boundary tori are the noncyclic p^2 cover of a boundary torus of M_i. If M_i is Seifert fibered then Theorem 3 gives such a cover $\widetilde{M}_i$ with at most $((20000c^2 + 200c)3^{100c+2} + 2)p^2$ sheets for all primes $p \geq 3$. We wish to find a prime p such that $p > D(100c)$. Bertrand's postulate

states that for any real number $x > 1$ there is a prime in the interval x to $2x$ (See [Hua82] Theorem 7.1 for a proof). In light of this theorem we may choose p to be less than $2D(100c)$. Then each geometric piece M_i has a cover $\widetilde{M_i}$ whose boundary components are the noncyclic p^2-sheeted covers of the boundary components of M_i.

From copies of these pieces we may assemble a cover $\widetilde{M}$ of M with at most $p^2 \cdot \mathrm{LCM}(\frac{m_0}{p^2}, \frac{m_1}{p^2}, \ldots, \frac{m_r}{p^2})$ sheets.

$$\begin{aligned} p^2 \cdot \mathrm{LCM}\left(\frac{m_0}{p^2}, \frac{m_1}{p^2}, \ldots, \frac{m_r}{p^2}\right) &\leq m_0 m_1 \cdots m_r / p^{2r-2} \\ &\leq (2D(100c))^{8c\left(1080000c^2 24^{100c}+2\right)} / D(100c)^{16c-2}. \end{aligned} \tag{60}$$

Thus we have completed the proof of Theorem 1.

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