Orthogonal Spectra

Last time: sequential spectra

Spaces \overset{\Sigma^\infty}{\rightsquigarrow} Spectra \quad \text{Quillen pair}

play nicely with homotopy theory

Spectra \overset{H}{\hookrightarrow} Ab

HA \overset{1}{\hookleftarrow} A

with \text{\textit{th space}} \ K(A,n)

H faithfully embeds Ab into Spectra

A ring is a monoid in \((\text{Ab}, \otimes, \mathbb{Z})\)

Need to define a symmetric monoidal product on \text{Sp}
Problem: there is a finite list of desirable properties for a category $C$ with $\text{ho}(C)$, the stable homotopy category.

Lewis proves you can only ever have 4 of 5 properties. Impossible to have all at once.

Remark: Actually, you can with $\infty$-cats.

Models of Spectra: a category $C$ that has 4 of 5 properties

- symmetric spectra
- orthogonal spectra
- EKMM spectra
- $\Gamma$-spaces
  - a spectrum $X$ is connective if $\pi_n X = 0$ for $n \leq 0$
  - $\Gamma$-spaces only give connective spectra
Def: An orthogonal $G_N$ spectrum $X$ consists of the following data:
- pointed spaces $X_n$ with $O(n)$-action
- maps $\sigma_n : X_n \wedge S^1 \to X_{n+1}$ such that the composite $\sigma_{n+m-1} \circ \ldots \circ \sigma_n$
  
  \[
  G \times O(n) \to O(n+m)
  \]
  
  \[
  X_n \wedge S^m \to X_{n+m}
  \]
  
  is $O(n) \times O(m) \leq O(n+m)$-equivariant $G \times O(n) \times O(m)$ $O(m) \wedge S^m$ if we think of $S^m$ as the 1-pt compactification of $\mathbb{R}^m$.

\[
\begin{bmatrix}
A_n & 0 \\
0 & A_m
\end{bmatrix}
\]

Def: A morphism of orthogonal spectra $f : X \to Y$ is a sequence of $G \times O(n)$-equivariant maps $X_n \overset{f_n}{\to} Y_n$ such that the following diagram commutes:

\[
\begin{array}{c}
X_n \wedge S^m & \overset{f_n \wedge \text{id}}{\longrightarrow} & Y_n \wedge S^m \\
\downarrow & & \downarrow \\
X_{n+m} & \overset{f_{n+m}}{\longrightarrow} & Y_{n+m}
\end{array}
\]
Examples: $n$th space $S^n = 1$-pt compactification of $\mathbb{R}^n$

$S^n \times S^m \rightarrow S^{n+m} \quad (v, w) \rightarrow [v \cup w]$ initial ring spectrum

Eilenberg–Maclane Spectra $HA$

If $A$ is a ring, $HA$ is a ring spectrum

$HA_n = A[S^n] = \left\{ \sum a_i \cdot s^i \mid a_i \in A, s_i \in S^n \right\}$

think $A[\mathbb{R}^n] \cup A \langle 00 \rangle$

Suspension Spectra: $\Sigma X$

$n$th space $S^n \wedge X$

$O(n) \uparrow$ trivial action

Def: An orthogonal ring spectrum $R$ is an orthogonal spectrum $R$ together with

"graded multiplication" $m_{n,m} : R_n \wedge R_m \rightarrow R_{n+m} \quad O(n) \times O(m)$ - equivariant

$in : S^n \rightarrow R_n \quad$ for $n \geq 0 \quad O(n)$ - equivariant

has associativity, unit properties
Associative
\[ R_n \wedge R_m \wedge R_p \xrightarrow{1 \times \mu} R_n \wedge R_{m+p} \]
\[ \downarrow \mu \downarrow \mu \]
\[ R_{n+m} \wedge R_p \xrightarrow{\mu} R_{n+m+p} \]

Unit
\[ R_0 \cong R_0 \wedge S^0 \xrightarrow{id \wedge i_0} R_0 \wedge R_0 \xrightarrow{\mu \circ i_0} R_n \]
(and the symmetric thing)

multiplicativity \[ \mu \circ i_n \circ i_m = i_{n+m} \]
\[ S^n \wedge S^m \xrightarrow{i_n \wedge i_m} R_n \wedge R_m \xrightarrow{\mu} R_{n+m} \]

Centrality
\[ R_m \wedge S^n \xrightarrow{id \wedge i_n} R_m \wedge R_n \xrightarrow{\mu} R_{m+n} \]
\[ \downarrow \kappa_{n,m} \]
\[ S^n \wedge R_m \xrightarrow{i_n \circ i_m} R_n \wedge R_m \xrightarrow{\mu} R_{n+m} \]

\[ \kappa_{n,m} = \begin{bmatrix} 0 & I_n \\ I_m \circ \phi & 0 \end{bmatrix} \]
Properties of Stable Homotopy Category:

$Ho(Sp)$ is the stable homotopy category:

- it is abelian
  - finite products $=$ finite coproducts
  - has all kernels ($= \text{fibers}$)
  - all cokernels ($= \text{cofibers}$)

- fiber seqs $A \to B \to C$ are cofiber seqs

- it is triangulated
  - associated w/ chain complexes
  - has a shift functor $\Sigma$
  - shift in the other direction $\Omega$
  - has a notion of "distinguished triangle"$
  \text{which yields long exact sequences}$

- it is closed symmetric monoidal
  - has a monoidal product $\wedge$
    - w/ unit $\mathbb{S}$
  - has an internal hom \(F(X,Y)\)
    - map \(X,Y\)

\[F(X \wedge Y, Z) \cong F(X, F(Y,Z))\]
Spaces $\xrightarrow{\Sigma^\infty} \text{Spectra}$

$\Omega^\infty X = \text{colim} \: \Omega^\infty X_n$

$\Sigma^\infty(\chi \rightarrow \Sigma^2 Y)$

$\text{Hom}_{\text{ch}_0(\mathbb{Z})}(A_\ast, B_\ast) = \bigoplus \text{degree } n \text{ homs } n + \mathbb{Z}$

$\text{Hom}_{\text{ho}(\text{Sp})}(X, Y) \text{ includes maps of degree } \neq 0$

$X \rightarrow \Sigma^2 Y$

$\chi_\ast S^1 \xrightarrow{\sigma} X_\ast \xrightarrow{\Sigma + \Omega} X_0 \xrightarrow{\tilde{\sigma}} \text{Map}(S^1, X_1)$

$\Omega^\infty X = \text{colim} \left( X_0 \xrightarrow{\tilde{\sigma}} \Omega X_1 \xrightarrow{\tilde{\sigma}} \Omega^2 X_2 \xrightarrow{\tilde{\sigma}} \cdots \right)$

$\pi_n \Omega^\infty X = \pi_n^S(X) = \pi_n(\Sigma^\infty X)$
Cory Malkiewicz has a good overview on his webpage.

Stefan Schwede - Lectures on Equivariant Stable homotopy theory

Birgit Richter - Commutative Ring Spectra

Handbook of Homotopy Theory (on nlab)

\[ G\text{-equivariant spectra should be indexed on } G\text{-reps} \]

let \( V \) be a \( G \)-rep, \( X \) on orthogonal spectrum

\[ X(V) = \text{lin}(\mathbb{R}^n, V) \wedge \bigwedge_{\mathcal{A}_n} X_n \]

Hom "groups" of \( G \)-spectra are Mackey functors