Cofibration Category

Yun Liu

March 17, 2019

1 Cofibration Category

Definition 1.1. A **cofibration category** is a category $\mathcal{C}$ with two classes of morphisms

$$(\mathcal{C}, cof, we)$$

called cofibrations and weak equivalences subject to the following axioms.

(C1) *Composition axiom*: the isomorphisms in $\mathcal{C}$ are weak equivalences and also cofibrations.

(C2) *Pushout axiom*: The pushout of a (trivial) cofibration always exists and is a (trivial) cofibration. For a cofibration $i : A \hookrightarrow B$ and $f : A \to Y$, there exists the pushout in $\mathcal{C}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, hook, "\bar{i}"] \\ B \arrow[r, "\bar{f}"] & B\bigcup\limits_{A} Y \end{tikzcd}$$

and $\bar{i}$ is a cofibration. Moreover,

1. if f is a weak equivalence, so is $\bar{f}$.
2. if i is a weak equivalence, so is $\bar{i}$.

(C3) *Factorization axiom*: Every map $f : A \to Y$ in $\mathcal{C}$ can be factor as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\ & X \arrow[ur, "g"', "\sim"] & \end{tikzcd}$$

where i is a cofibration and g is a weak equivalence.

(C4) *Axiom on fibrant models*: For each object X in C there is a trivial cofibration $X \overset{\sim}{\hookrightarrow} RX$ where RX is fibrant in $\mathcal{C}$. We call $X \overset{\sim}{\hookrightarrow} RX$ a **fibrant model** of X.

Definition 1.2. An object R in a cofibration category is a **fibrant model** (or fibrant) if each trivial cofibration $i : R \overset{\sim}{\hookrightarrow} Q$ in $\mathcal{C}$ admits a retraction $r \circ i : Q \to R, ir = \mathrm{Id}_R$.

Let Ob_f be a class of fibrant models in $\mathcal{C}$ which is sufficiently large, i.e. each object in $\mathcal{C}$ has a fibrant model in Ob_f. The cofibrations and weak equivalences in Ob_f are inherited from the ones in $\mathcal{C}$ and it's easy to see that axiom **(C1)**, **(C2)** and **(C4)** are satisfied, if furthermore (C3) is satisfied, then Ob_f is a cofibration category.

(C2) can be verified in this way: For any object A, Y in $\mathcal{C}_f$ and $f : A \to Y$, we have a factorization $f = g \circ i$ in $\mathcal{C}$, and we consider a pushout with $IB \in Ob\,(\mathcal{C}_f)$,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[dd, hook, "\bar{j}"', "\wr"] \\ X \arrow[ur, "\sim", "g"'] \arrow[d, hook, "j"', "\wr"] & \\ IX \arrow[r, "\sim", "\bar{g}"'] & IX \bigcup_X Y \arrow[uu, dashed, bend right, "r"', "\wr"] \end{tikzcd}$$

then $\bar{j}$ is a trivial cofibration and $\bar{g}$ is a weak equivalence by pushout axiom, and since Y is a fibrant model, there is retraction $r : IX \bigcup_X Y \xrightarrow{\sim} Y$, then we get a desired factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, hook, "j \circ i"'] & & Y \\ & IX \arrow[ur, "r \circ \bar{g}"', "\sim"] & \end{tikzcd}$$

Definition 1.3. If $\mathcal{C}$ has an initial object $\emptyset$, an object X in $\mathcal{C}$ is **$\emptyset$-cofibrant** if $\emptyset \hookrightarrow X$ is a cofibration.

Claim 1.1. Let $\mathcal{C}_c$ be the full subcategory of $\mathcal{C}$ consisting of cofibrant objects with cofibrations and weak equivalences inherited from the ones in $\mathcal{C}$. Then $\mathcal{C}_c$ is a cofibration category.

Proof. Axiom **(C1)** is easy to see. For axiom **(C2)**, it suffices to show $B\bigcup\limits_A Y$ is cofibrant. Consider a pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \arrow[d, hook, "\bar{i}"] \\ B \arrow[r, "\bar{f}"] & B\bigcup\limits_A Y \end{tikzcd}$$

Since Y is cofibrant, $B\bigcup\limits_A Y$ is also cofibrant, hence $B\bigcup\limits_A Y$ is in $\mathcal{C}_c$.

For axiom **(C3)**, it suffices to see that for every map $f : A \to Y$ in $\mathcal{C}_c$, f can be factor in $\mathcal{C}$ as

$$\begin{tikzcd} A \arrow[rr, hook, "f"] \arrow[dr, hook, "i"'] & & Y \\ & X \arrow[ur, "\sim", "g"'] & \end{tikzcd}$$

and since i is a cofibration, X is cofibrant, so this is a factorization in $\mathcal{C}_c$.

For axiom **(C4)**, note ant fibrant model of of a $\emptyset$-cofibrant object is $\emptyset$-cofibrant. $\square$

Remark 1.1. The notion of $\emptyset$-cofibrant is not dual to the notion of fibrant in **(C4)**, and it depends on the existence of the initial object $\emptyset$.

1.1 Mapping Cylinder and Homotopy

Let $i : A \hookrightarrow B$ be a cofibration, consider the pushout

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, hook, "i"'] & B \arrow[d, hook] \arrow[ddr, bend left, "1"] & \\ B \arrow[r, hook] \arrow[drr, bend right, "1"'] & B\bigcup\limits_A B \arrow[dr, dotted, "\varphi"] & \\ & & B \end{tikzcd}$$

which induces a map $\varphi : B\bigcup_A B \to B$ called **fold map**. We have a factorization of φ

$$B\bigcup_A B \overset{j}{\hookrightarrow} Z = I_B A \xrightarrow[\sim]{p} B$$

Then $Z = I_B A$ is called a **relative cylinder** on $i : A \hookrightarrow B$. Note the maps $i_\varepsilon : B \to Z, \varepsilon = 0, 1$ are trivial fibration.

If X is a fibrant object, any two maps $\alpha, \beta : B \to X$ are homotopic relative (or under) B if there is a commutative diagram

$$\begin{tikzcd} B\bigcup_A B \arrow[rr, "{(\alpha,\beta)}"] \arrow[dr, hook, "j"'] & & X \\ & Z \arrow[ur, "H"'] & \end{tikzcd}$$

where $Z = I_B A$ is a relative cylinder on $i : A \hookrightarrow B$. H is called a **homotopy** from α to β. This is an equivalence relation.

For a $\emptyset$-cofibrant object A there exists the sum $A + Y = A\bigcup_\emptyset Y$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\ A \arrow[r] & A + Y \end{tikzcd}$$

where $Y \to A + Y$ is a cofibration. Also $A \to A + Y$ is cofibration provided Y is $\emptyset$-cofibrant.

The **mapping cylinder** Z_f of $f : A \to Y$ is defined by a factorization of the map $(1_A, f) : A + Y \to Y$

$$A + Y \hookrightarrow Z_f \xrightarrow[q]{\sim} Y$$

If Y is cofibrant this yields a factorization $f = qi_1$

$$A \overset{i_1}{\hookrightarrow} Z_f \xrightarrow[q]{\sim} Y$$

where q is a retraction of $i_0 : Y \hookrightarrow A + Y \hookrightarrow Z_f$ and $i_1 : A \hookrightarrow A + Y \hookrightarrow Z_f$.

Moreover, we can use the cylinder $Z = I_\emptyset A$ to construct a mapping cylinder via pushout, not in the following diagram, the upper and lower squares are both pushouts.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, hook] \arrow[ddr, bend left, "1_Y"] & \\
A+A \arrow[r, "f+1"] \arrow[d, hook, "{(i_0,i_1)}"'] & Y+A \arrow[d, hook, "{(\bar{i}_0,\bar{i}_1)}"] \arrow[dr, bend left, "{(1_Y,f)}"] & \\
Z \arrow[r, "\pi"] \arrow[d, "p"', "\wr"] & Z_f \arrow[dr, dotted, "q"] & \\
A \arrow[rr, "f"] & & Y
\end{tikzcd}$$

Note since p is a weak equivalence, $A \hookrightarrow Z$ is a trivial fibration, thus $Y \hookrightarrow Z_f$ is a trivial fibration, so q is a weak equivalence.

Lemma 1.1. *Let $\mathcal{C}$ be a cofibration category, then axiom **(C2)(1)**, **(C1)** and **(C3)** imply **(C2)(2)**. If all objects in $\mathcal{C}$ are cofibrant then axiom **(C2)(2)**, **(C1)** and **(C3)** imply **(C2)(1)**.*

Proof. We consider the pushout diagram

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[rr, bend left, "f"] \arrow[d, hook, "j"'] & Z_f \arrow[r, "g", "\sim"'] \arrow[d, hook, "j'"] & Y \arrow[d, hook, "\bar{j}"] \\
B \arrow[r, hook, "\bar{i}"'] \arrow[rr, bend right, "\bar{f}"'] & \bar{Z}_f \arrow[r, "\bar{g}"'] & Y
\end{tikzcd}$$

where $f = g \circ i$ is the factorization given by axiom **(C3)**. If j is a weak equivalence, then j' is a weak equivalence by **(C2)(1)**, and similarly $\bar{g}$ is a weak equivalence, so $\bar{j}$ is a weak equivalence.

If every object in $\mathcal{C}$ is cofibrant, and f is weak equivalence then i is a weak equivalence, so by axiom **(C2)(2)**, $\bar{i}$ is a weak equivalence. Now we need to show $\bar{g}$ is a weak equivalence. Since X is cofibrant, g admits a retraction r, so consider the following pushout diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "j"'] & Y \arrow[r, "r"] \arrow[rr, bend left, "1"] \arrow[d, hook, "\bar{j}"] & X \arrow[r, "g", "\sim"'] \arrow[d, hook, "j'"] & Y \arrow[d, hook, "\bar{j}"] \\
B \arrow[r, "\bar{f}"'] & \bar{Y} \arrow[r, "\bar{r}"'] \arrow[rr, bend right, "1"'] & X \arrow[r, "\bar{g}"'] & \bar{Y}
\end{tikzcd}$$

we see that $\bar{g} \circ \bar{i} = 1_{\bar{Y}}$. □

1.2 Homotopy Pushout

A commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

in a cofibration category is a **homotopy pushout** if for some factorization $A \hookrightarrow W \xrightarrow{\sim} B$ the induced map $W \bigcup_A X \to Y$ is a weak equivalence,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, hook] \\ W \arrow[r] \arrow[d, "h"', "\wr"] & W \bigcup_A X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

This implies for any factorization $A \hookrightarrow W \xrightarrow{\sim} B$ the induced map $W \bigcup_A X \to Y$ is a weak equivalence.

Exercise 1.1. Given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[d] & Y \arrow[d] \\ C \arrow[r] & Z \end{tikzcd}$$

if the upper square is a (homotopy) pushout, then the outer square is a (homotopy) pushout if and only if the lower square is a (homotopy) pushout.

Proof. The second one follows from the following lemma. □

Lemma 1.2. *Consider a commutative diagram*

$$\begin{tikzcd} X_0 \arrow[d, "\alpha"'] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\gamma"] & X_1 \arrow[d, "\beta"] \\ Y_0 & Y \arrow[l, "f'"'] \arrow[r, "g'"] & Y_1 \end{tikzcd}$$

where in each row one of the maps is a fibration, then the pushouts of the rows exist and we have a map

$$\alpha \cup \beta : X_0 \bigcup_X X_1 \to Y_0 \bigcup_Y Y_1$$

1. *Assume that α, β, γ and the induced map $(g, \beta) : Y \bigcup_X X_1 \to Y_1$ are cofibrations, then so is $\alpha \cup \beta$.*

2. *Assume that α, β, γ are weak equivalences, then so is $\alpha \cup \beta$.*

1.3 Functors Between Cofibration Categories

Let $\mathcal{C}, \mathcal{K}$ be cofibration categories and let $\alpha : \mathcal{C} \to \mathcal{K}$ be a functor.

1. The functor α is **based** if $\mathcal{C}$ and $\mathcal{K}$ have an initial object (denoted by $*$) with $\alpha(*) = *$.

2. The functor α preserves weak equivalences if α carries a weak equivalence in $\mathcal{C}$ to a weak equivalence in $\mathcal{K}$.

3. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\ B \arrow[r] & Y = B \bigcup_A X \end{tikzcd}$$

be a pushout diagram. We say that α is **compatible with the pushout** $B \bigcup_A X$ if the induced diagram

$$\begin{tikzcd} \alpha A \arrow[r, "f"] \arrow[d, "g"'] & \alpha X \arrow[d] \\ \alpha B \arrow[r] & \alpha Y = \alpha \left(B \bigcup_A X \right) \end{tikzcd}$$

is a homotopy pushout in $\mathcal{K}$.

4. We call α a **model functor** if α preserves weak equivalences and if α is compatible with all pushouts. (Hence a model functor carries a homotopy pushout in $\mathcal{C}$ to homotopy pushout in $\mathcal{K}$.

Remark 1.2. A based model functor α is compatible with most of the constructions in a cofibration category. In general, we do not assume that a model functor carries a cofibration in $\mathcal{C}$ to a cofibration in $\mathcal{K}$.

Proposition 1.1. *Let $\alpha, \beta : \mathcal{C} \to \mathcal{K}$ be functors between cofibration categories which are natural weak equivalent (finite chain of weak equivalences). If α preserves weak equivalences, then so does β. If α is compatible with homotopy pushout, then so is β. Hence if α is a model functor then so is β.*

Corollary 1.1. *If $\mathcal{C}$ is a cofibration category and $\alpha : \mathcal{C} \to \mathcal{C}$ is naturally weak equivalent to the identical functor, then α is a model functor.*

Example 1.1. Let **Top** the the cofibration category of topological spaces, then the singular realization functor yields a functor

$$|S| : \mathbf{Top} \to \mathbf{Top}$$

which carries a space X to the CW complex $|SX|$, which is naturally weak equivalent to the identical functor.

2 Homotopy Theory of a Cofibration Category

2.1 Sets of Homotopy Classes

For a cofibration $Y \subseteq X$ and a map $uY \to U$ let

$$\mathrm{Hom}\,(X,U)^u$$

be the set of all maps $f : X \to U$ in $\mathcal{C}$ which $f|_Y = u$. We say f is an **extension** of u. On this set we have the homotopy relation relative Y which we denote by '$\simeq$rel Y' .

Proposition 2.1. *Let U be fibrant, Then all cylinders on $Y \subset X$ define the same homotopy relation relative Y on*

$$Hom\,(X,U)^u\,.$$

Moreover, the homotopy relation relative Y is an equivalence relation.

Thus if U is fibrant, we have the set

$$[X,U]^Y = [X,U]^u = \mathrm{Hom}\,(X,U)^u \,/\simeq \mathrm{rel}Y$$

of homotopy classes.

In particular, for an initial object $\emptyset$ in $\mathcal{C}$, let

$$[X,U] = [X,U]^{\emptyset} = \mathrm{Hom}\,(X,U) \,/\simeq \mathrm{rel}Y$$

be the set of homotopy classes of maps from X to U.

2.2 The Homotopy Category of fibrant and cofibrant objects

Let $\mathcal{C}$ be a cofibration category with initial object $\emptyset$, then we have the full subcategories

$$\mathcal{C}_{cf} \subset \mathcal{C}_c \subset \mathcal{C}$$

where $\mathcal{C}_{cf}$ consists of objects which are both fibrant and cofibrant, and $\mathcal{C}_c$ is the category of cofibrant objects. Note that $\mathcal{C}_c$ is a cofibration category, but $\mathcal{C}_{cf}$ in general is not.

Lemma 2.1. *Homotopy relative $\emptyset$ is a natural equivalence relation on the morphism set of $\mathcal{C}_{cf}$.*

We thus have the homotopy category

$$\mathcal{C}_{cf}/\simeq = \mathcal{C}_{cf}/\left(\simeq \mathrm{rel}\emptyset\right).$$

Let $\mathrm{Ho}\left(\mathcal{C}\right)$ be the localization of $\mathcal{C}$ with respect to the given class of weak equivalences in $\mathcal{C}$.

Proposition 2.2. *We have equivalences of homotopy categories*

$$Ho\left(\mathcal{C}_{cf}\right) \xrightarrow[\sim]{i} Ho\left(\mathcal{C}_c\right) \xrightarrow[\sim]{j} Ho\left(\mathcal{C}\right)$$

3 Examples of Cofibration Categories

3.1 Topological Spaces

Theorem 3.1. *The category* **Top** *of topological spaces with the class of cofibrations which are maps that has homotopy extension property with respect to any space in* **Top** *and the class of weak equivalences being the maps which are homotopy equivalences is a cofibration category in which all objects are fibrant and cofibrant.*

Theorem 3.2. *(Dual) The category* **Top** *of topological spaces with the class of fibrations which are maps that has homotopy lifting property with respect to any space in* **Top** *and the class of weak equivalences being the maps which are homotopy equivalences is a fibration category in which all objects are fibrant and cofibrant.*

Theorem 3.3. *The category* **Top** *of topological spaces with the class of cofibrations which are inclusions $B \subset A$ for which A is given by a well-ordered succession of attaching cells to B, and weak equivalences are weak homotopy equivalences is a cofibration category in which all objects are fibrant models. CW complexes are cofibrant objects and all cofibrant objects are CW spaces(homotopy equivalent to a CW complex).*

Example 3.1. The category $\mathcal{C}$ has objects (X, N_X) where X is a well-pointed path-connected CW space and N_X is a perfect and normal subgroup of $\pi_1(X)$, and morphisms are maps $f : (X, N_X) \to (Y, N_Y)$ which are basepoint preserving maps $f : X \to Y$ in $\mathbf{Top}_*$ with $f_*(N_X) \subset N_Y$.

The cofibrations are maps in $\mathcal{C}$ which are cofibrations in **Top**.

Weak equivalences are maps in $\mathcal{C}$ which induces isomorphisms

$$f_* : \pi_1(X)/N_X \cong \pi_1(Y)/N_Y$$

and

$$f_* : \hat{H}_*(X, f_*^* q^* \mathcal{L}) \cong \hat{H}_*(Y, q^* \mathcal{L})$$

here $\hat{H}_*$ denote homology with local coefficients. The modules $q^*\mathcal{L}$ and $f_*^* q^* \mathcal{L}$ are lifted by $q : \pi_1(Y) \to \pi_1(Y)/N_Y$ and by $f_* : \pi_1(X) \to \pi_1(Y)$ respectively.

Theorem 3.4. *The structure above is a cofibration category where a fibrant model of* (X, N_X) *is given by Quillen's* $(+)$*-construction.*

3.2 The Category of Chain Complexes

Weak equivalences: $f : V \to V'$ which induces isomorphisms on homology.

Cofibrations: injective chain maps with free (projective) cokernel.

Theorem 3.5. *The category of bounded chain complexes with the above structure is a cofibration category.*

3.3 The Category of Chain Algebras

Weak equivalences: $f : V \to V'$ which induces isomorphisms on homology.

Cofibrations: $f : A \to B$ such that there is a submodule V of B such that V is free and $A \coprod T(V) \to B$ is an isomorphism of algebras.

Theorem 3.6. *Let k be a principal ideal domain. The category* $\mathbf{DA}_k$ *of chain algebra over k with the above structure is a cofibration category.*

3.4 The Category of Chain Lie Algebras

Weak equivalences: $f : A \to B$ which induces isomorphisms on homology.

Cofibrations: $f : A \to B$ such that there is a submodule V of B such that V is free and $A \coprod T(V) \to B$ is an isomorphism of Lie algebras.

Theorem 3.7. *Let k be a principal ideal domain. The category* $\mathbf{LA}$ *of chain Lie algebras with the above structure is a cofibration category.*

References

[1] H. J. Baues, Algebraic homotopy. Cambridge University Press, Cambridge, New York, 1989, ISBN 0-521- 33376-8. https://web.math.rochester.edu/people/faculty/doug/otherpapers/baues4.pdf