1 Talk 1: cotorsion pairs and abelian model categories

The goal of this talk is to introduce the notion of cotorsion pairs in an abelian category, and go over a result by Mark Hovey ([Hov02]) that relates cotorsion pairs to suitably algebraic model structures one can define on the abelian category.

We start by defining cotorsion pairs. These were introduced in the late 70's by Salce, and became more widely known in the 90's when Enochs and coauthors used them to prove the flat cover conjecture: namely, that for any ring R, all R-modules admit a flat cover. The definition is as follows.

Definition 1.1. A *cotorsion pair* in an abelian category $\mathcal{A}$ is a pair $(\mathcal{P}, \mathcal{I})$ consisting of two classes of objects of $\mathcal{A}$ that are the orthogonal complement of each other with respect to the Ext^1 functor. More explicitly,

(i) $P \in \mathcal{P}$ if and only if $\mathrm{Ext}^1(P, I) = 0$ for every $I \in \mathcal{I}$, and

(ii) $I \in \mathcal{I}$ if and only if $\mathrm{Ext}^1(P, I) = 0$ for every $P \in \mathcal{P}$.

Note that cotorsion pairs provide a generalization of injective and projective objects in an abelian category; indeed, an object P is projective precisely when the functor $\mathrm{Hom}(P, -)$ is exact, which is equivalent to requiring that $\mathrm{Ext}^1(P, A) = 0$ for every $A \in \mathcal{A}$. Thus, if Proj denotes the class of projective objects in $\mathcal{A}$, we see that $(\mathcal{P}, \mathcal{A})$ is a cotorsion pair. Similarly, if Inj denotes the class of injective objects, then $(\mathcal{A}, \mathrm{Inj})$ is a cotorsion pair.

Borrowing motivation from the case of injectives and projectives, it is of interest to know when a cotorsion pair provides *resolutions* for any given object.

Definition 1.2. Let $(\mathcal{P}, \mathcal{I})$ be a cotorsion pair in an abelian category $\mathcal{A}$. We say the cotorsion pair is *complete* if any object A in $\mathcal{A}$ can be resolved as

$$0 \to A \to I \to P \to 0$$

for some $I \in \mathcal{I}$, $P \in \mathcal{P}$, and as

$$0 \to I' \to P' \to A \to 0$$

for some $I' \in \mathcal{I}$, $P' \in \mathcal{P}$.

Furthermore, we will say the pair is *functorially complete* if any map $f\colon A \to B$ can be lifted to a map between some resolutions of A and of B

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & I_A \arrow[r] \arrow[d] & P_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] & I_B \arrow[r] & P_B \arrow[r] & 0
\end{tikzcd}$$

and similarly for the other type of resolutions.

Most of the cotorsion pairs that arise in nature are (or can be chosen to be) functorially complete.

As an example, we can see that the cotorsion pair $(\mathrm{Proj}, \mathcal{A})$ is complete precisely if $\mathcal{A}$ has enough projectives.

Definition 1.3. A cotorsion pair $(\mathcal{P}, \mathcal{I})$ is *hereditary* if the class $\mathcal{P}$ is closed under kernels of epimorphisms, or equivalently, if $\mathcal{I}$ is closed under cokernels of monomorphisms. More explicitly, if whenever $f : A \to B$ is an epimorphism for $A, B \in \mathcal{P}$, we have $\ker f \in \mathcal{P}$, and similarly for $\mathcal{I}$.

As an example, the cotorsion pair $(\mathcal{P}, \mathcal{A})$ is hereditary: given an epimorphism $f : A \to B$, we can consider the exact sequence

$$0 \to \ker f \to A \xrightarrow{f} B \to 0.$$

Since B is projective, this sequence splits, which makes $\ker f$ a direct summand of A (which is projective), and therefore projective.

Remark 1.4. Those of you who know a bit about algebraic K-theory may have seen this condition before: the Gillet-Waldhausen theorem, which states that if $\mathcal{C}$ is an exact category, then

$$K(\mathcal{C}) \simeq K(\mathsf{Ch}^{\mathsf{b}}(\mathcal{C}))$$

requires $\mathcal{C}$ to be closed under kernels of epimorphisms.

We now jump to model categories, which, as we know, are a setting in which to do homotopy theory. For the sake of completeness, we briefly recall the definition.

Definition 1.5. A *model category* is a category $\mathcal{C}$ with finite limits and colimits, together with three classes of maps in $\mathcal{C}$ called cofibrations, fibrations, and weak equivalences, that are closed under composition, contain the identity maps, and satisfy the following axioms:

1. **(2-out-of-3)** if 2 out of the 3 maps f, g and gf are weak equivalences, so is the third,

2. **(retracts)** cofibrations, fibrations and weak equivalences are closed under retracts,

3. **(lifts)** any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r] & D \end{tikzcd}$$

   where either i is a cofibration and p a trivial fibration, or i a trivial cofibration and p a fibration, there exists a lift $C \to B$,

4. **(factorizations)** any map f can be factored as $f = pi$, for some cofibration i and trivial fibration p, and as $f = qj$ for some trivial cofibration j and fibration q.

So, how do model categories relate to cotorsion pairs? Let $\mathcal{A}$ be an abelian category, and consider a cotorsion pair $(\mathcal{P}, \mathcal{I})$ in $\mathcal{A}$. Given an exact sequence

$$0 \to A \xrightarrow{i} B \to P \to 0$$

where $P \in \mathcal{P}$, we have that, for any $I \in \mathcal{I}$, the map $\mathcal{A}(B, I) \to \mathcal{A}(A, I)$ is surjective (since $\mathrm{Ext}^1(P, I) = 0$). That means for any diagram as below

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & I \arrow[d] \\ B \arrow[r] & 0 \end{tikzcd}$$

there exists a lift $B \to I$. This has a flavor of the lifting axiom in model categories, and if we imagine i to be a trivial cofibration, we would be saying something like "every object in $\mathcal{I}$ is fibrant".

Let's make this more precise.

Definition 1.6. An *abelian model category* is an abelian category $\mathcal{A}$ equipped with a model structure in which

1. cofibrations are the monomorphisms with cofibrant cokernel,

2. fibrations are the epimorphisms with fibrant kernel.

It is easy to show that, in this case, trivial cofibrations are monomorphisms with an acyclic cofibrant cokernel, and trivial fibrations are epimorphisms with an acyclic fibrant kernel.

Note that this is not circular: we are not defining the model structure by these rules, instead, we start with a model structure (and thus, a notion of cofibrations, fibrations, and classes of cofibrant and fibrant objects already determined), and require that these conditions be satisfied.

Also note that abelian model categories are "suitably algebraic" in the sense that the cofibrations and fibrations preserve the structure of the underlying abelian category, in this specific way.

Hovey showed in [Hov02] that any algebraic model category determines two complete cotorsion pairs, in the following way:

Theorem 1.7. *Let $\mathcal{A}$ be an abelian model category, and denote by $\mathcal{C}, \mathcal{F}, \mathcal{Z}$ the classes of cofibrant, fibrant, and acyclic objects, respectively. Then $(\mathcal{C} \cap \mathcal{Z}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{Z})$ are complete cotorsion pairs.*

Proof. We sketch the proof for the case of $(\mathcal{C}, \mathcal{F} \cap \mathcal{Z})$. Let C be a cofibrant object, and F an acyclic fibrant one; we first wish to show that $\mathrm{Ext}^1(C, F) = 0$. Now, remember that the elements of $\mathrm{Ext}^1(C, F)$ can be viewed as the isomorphism classes of extensions of C by F. Then, we need to show that all such extensions are isomorphic to the trivial extension (given by the direct sum), which is equivalent to proving that any extension

$$0 \to F \xrightarrow{i} A \to C \to 0$$

splits.

Since i is a monomorphism with cofibrant cokernel, it's a cofibration, and so the diagram

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d, "i"'] & F \arrow[d] \\ A \arrow[r] & 0 \end{tikzcd}$$

admits a lift, which provides a splitting for our sequence.

Now, suppose that $\mathrm{Ext}^1(A, F) = 0$ for all acyclic fibrant F; we must show that $A \in \mathcal{C}$. Let $g : B \to C$ be a trivial fibration, that is, g fits into an exact sequence

$$0 \to \ker g \to B \xrightarrow{g} C \to 0$$

where $\ker g \in \mathcal{F} \cap \mathcal{Z}$, and consider any diagram of the form

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r] & C \end{tikzcd}$$

Applying the functor $\mathcal{A}(A, -)$ to the above sequence, we see that $\mathcal{A}(A, g) : \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ is an epimorphism, and so the above square must have a lift. This shows that $0 \to A$ is a cofibration, and thus $A \in \mathcal{C}$. Dually, one shows that if $\mathrm{Ext}^1(C, B) = 0$ for every $C \in \mathcal{C}$, then $B \in \mathcal{F} \cap \mathcal{Z}$.

To prove that the cotorsion pair is complete, we consider an object $A \in \mathcal{A}$ and try to find a resolution

$$0 \to F \to C \to A \to 0$$

for some $F \in \mathcal{F} \cap \mathcal{Z}$, $C \in \mathcal{C}$. Note that this would be trivial if A was cofibrant, since any class in a cotorsion pair contains the zero object. If not, we find a cofibrant replacement: the map $0 \to A$ factors as $0 \xrightarrow{i} \overline{A} \xrightarrow{p} A$ for some cofibration i and trivial fibration p. Then $\overline{A} \in \mathcal{C}$, and $\ker p \in \mathcal{F} \cap \mathcal{Z}$, so A can be resolved by

$$0 \to \ker p \to \overline{A} \xrightarrow{p} A \to 0$$

Dually, we find an injective resolution. $\square$

This is a fun result, where we can see how the model structure comes to our aid to prove, quite easily, the required conditions for the cotorsion pair. More interesting, perhaps, is the fact that this result admits a converse: every abelian model structure is determined by a compatible pair of complete cotorsion pairs.

Theorem 1.8. *Let $\mathcal{C}, \mathcal{F}$ and $\mathcal{Z}$ be classes of objects in a bicomplete abelian category $\mathcal{A}$, such that $(\mathcal{C}, \mathcal{F} \cap \mathcal{Z})$ and $(\mathcal{C} \cap \mathcal{Z}, \mathcal{F})$ are complete cotorsion pairs. Suppose that, in addition, $\mathcal{Z}$ is closed under retracts and, whenever 2 out of 3 terms in a short exact sequence are in $\mathcal{Z}$, then so is the third. Then there exists a unique abelian model structure on $\mathcal{A}$ having $\mathcal{C}$ as the class of cofibrant objects, $\mathcal{F}$ as the class of fibrant objects, and $\mathcal{Z}$ as the class of acyclic objects.*

This proof is much, much more involved than the previous one for the opposite direction. Interestingly, properties that are usually hard to prove when one wishes to determine a model structure, like the lifting and factorization axioms, are not so hard to show, and the main difficulty arises from defining the weak equivalences and proving the 2-out-of-3 axiom.

Remark 1.9. How are weak equivalences defined?

Suppose that f is a weak equivalence in our model structure. Then, we can factor f as $f = pi$ where p is a trivial fibration and i a cofibration. But weak equivalences have 2-out-of-3, so i must actually be a trivial cofibration.

We use this to define weak equivalences: they are precisely the maps that can be factores as a trivial cofibration (that is, a monomorphism with cokernel in $\mathcal{C} \cap \mathcal{Z}$) followed by a trivial fibration (that is, an epimorphism with kernel in $\mathcal{F} \cap \mathcal{Z}$).

Example 1.10. Let R be a quasi-Frobenius ring (that is, a ring such that projective modules and injective modules coincide). Then, one can take $\mathcal{C} = \mathcal{F} = R-$mod, and $\mathcal{Z} = \text{Proj} = \text{Inj}$; then, $(\mathcal{C}, \mathcal{F} \cap \mathcal{Z}) = (R - mod, \text{Inj})$ and $(\mathcal{C} \cap \mathcal{Z}, \mathcal{F}) = (\text{Proj}, R - mod)$ are complete cotorsion pairs, and induce a model category structure on $R-$mod whose homotopy category is known in representation theory as the *stable category of $R-$modules.*

Example 1.11. Let R be any ring, and consider the abelian category $\mathsf{Ch}(R)$ of unbounded chain complexes in R. If we take $\mathcal{C} = \mathsf{Ch}(R), \mathcal{Z}$ to be the class of exact complexes, and $\mathcal{F}$ the class of DG-injective complexes (exact + DG-injective = categorical injective), then $(\mathcal{C}, \mathcal{F} \cap \mathcal{Z}) = (\mathsf{Ch}(R), \text{Inj})$ and $(\mathcal{C} \cap \mathcal{Z}, \mathcal{F}) = (\text{exact}, DG - \text{injective})$ are complete cotorsion pairs, and they determine the usual injective model structure, where cofibrations are degreewise monomorphisms, and weak equivalences are quasi-isomorphisms.

One can recover the projective model structure from a similar pair of cotorsion pairs.

References

[Hov02] M. Hovey, *Cotorsion pairs, model category structures and representation theory*, Mathematische Zeitschrift, Springer-Verlag **241**: 553–592 (2002).