Eilenberg–MacLane Spectra as Thom Spectra

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1 Introduction

The goal of these two lectures is to prove a theorem, originally due to Bökstedt:

Theorem 1.1 (Bökstedt)**.**

$$\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x], \ \ |x| = 2$$

This was originally proved by a tedious spectral sequences argument, but that's not the approach we'll take. Instead, we'll take advantage of two different theorems.

Theorem 1.2 (Hopkins–Mahowald)**.** *There is an equivalence of $\mathbb{E}_2$-ring spectra*

$$H\mathbb{F}_p \simeq Mf_p,$$

where $f_p \colon \Omega^2 S^3 \to B\,GL_1(\widehat{\mathbb{S}_p})$ is the map determined by $1-p \in \pi_1 B\,GL_1(\widehat{\mathbb{S}_p}) \simeq \mathbb{Z}_p^\times$.

Theorem 1.3 (Blumberg–Cohen–Schlichtkrull)**.** *Let $f \colon X \to B\,GL_1(R)$ be an $\mathbb{E}_2$-map of spaces. Assume that the $\mathbb{E}_2$-structure on Mf extends to an $\mathbb{E}_3$-structure. Then there is an equivalence of $\mathbb{E}_1$-R-module spectra*

$$\mathrm{THH}(Mf/R) \simeq Mf \otimes BX_+$$

Proof of Theorem 1.1. We compute only $\mathrm{THH}(\mathbb{F}_p)$. The computation of $\mathrm{THH}(\mathbb{Z})$ is similar. By **??**, it suffices to compute $\mathrm{THH}(Mf_p)$. By Theorem 1.3,

$$\mathrm{THH}(H\mathbb{F}_p) \simeq \mathrm{THH}(Mf_p) \simeq Mf_p \otimes \Sigma^\infty_+ B(\Omega^2 S^3) \simeq Mf_p \otimes \Sigma^\infty_+ \Omega S^3 \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3.$$

So as a spectrum, $\mathrm{THH}(H\mathbb{F}_p) \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3_+$. The homotopy of this spectrum is (by definition!) the $\mathbb{F}_p$-homology of ΩS^3, which can be calculated by the Serre spectral sequence associated to the fibration

$$\Omega S^3 \to \mathrm{Map}([0,1], S^3) \to S^3.$$

$$E^2_{s,t} = H_s(S^3; H_t(\Omega S^3; \mathbb{F}_p)) \implies H_{s+t}(\mathrm{Map}([0,1], S^3); \mathbb{F}_p) = 0.$$

I leave this as an exercise (see Hatcher's spectral sequences book). □

Last time we focused on the Hopkins–Mahowald theorem, and constructed some Thom spectra. This time, we'll focus on the Blumberg–Cohen–Schlichtkrull theorem and use it to do some computations. But first, some reminders on Thom spectra.

2 Thom Spectra

Let R be an $\mathbb{E}_1$-ring spectrum.

Definition 2.1. Given a local system of R-modules, $f\colon X \to B\,GL_1(R)$, the associated Thom spectrum is

$$Mf = \operatorname{colim}(i \circ f) = \operatorname{colim}(X \xrightarrow{f} B\,GL_1(R) \xrightarrow{i} R\text{-}\mathbf{Mod}).$$

If f is an $\mathbb{E}_n$-map for some n, then Mf is an $\mathbb{E}_n$-R-module.

Here $GL_1(R)$ is defined by the homotopy pullback of spaces

$$\begin{tikzcd} GL_1(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Omega^\infty(R) \arrow[d] \\ (\pi_0 R)^\times \arrow[r] & \pi_0 R \end{tikzcd}$$

This is an $\mathbb{E}_1$-algebra in spaces, and if R is an E_∞-ring spectrum, then $GL_1(R)$ is an $\mathbb{E}_\infty$-algebra in spaces.

Example 2.2. If $X = \Omega^2\Sigma^2 Y$ for some space Y (i.e. X is the free $\mathbb{E}_2$-algebra on Y), then $\mathbb{E}_2$-maps $\Omega^2\Sigma^2 Y \to B\,GL_1(R)$ are in bijection with maps $Y \to B\,GL_1(R)$, by the universal property of free $\mathbb{E}_2$-algebras.

In particular, if $Y = S^1$, to define a map $\Omega^2 S^3 \to B\,GL_1(R)$, it suffices to pick an element in $\pi_1 B\,GL_1(R) \cong \pi_0\,GL_1(R) \cong \pi_0(R)^\times$.

This is how we will construct $H\mathbb{F}_p$ as a Thom spectrum.

$$H\mathbb{F}_p \simeq M(\Omega^2 S^3 \xrightarrow{f_p} B\,GL_1(\widehat{\mathbb{S}_p})),$$

where the map f_p is determined by $1 - p \in \pi_1 B\,GL_1(\widehat{\mathbb{S}_p}) \cong \mathbb{Z}_p^\times$.

Example 2.3.

$$H\mathbb{Z}_p \simeq M(\tau_{\geq 2}\Omega^2 S^3 \to \Omega^2 S^3 \xrightarrow{f_p} B\,GL_1(\widehat{\mathbb{S}_p}))$$

Example 2.4.

$$H\mathbb{Z} \simeq M\left(\tau_{\geq 2}\Omega^2 S^3 \xrightarrow{\prod_p f_p} \prod_{p \text{ prime}} B\,GL_1(\widehat{\mathbb{S}_p}) \to B\,GL_1(\mathbb{S})\right)$$

Lemma 2.5. *Let* R *be an* $\mathbb{E}_n$*-ring spectrum and let* S *be an* $\mathbb{E}_n$*-*R*-algebra. Let* $f\colon X \to B\,GL_1(R)$ *be a map of* $\mathbb{E}_n$ *spaces. Then there is an equivalence of* $\mathbb{E}_n$ *rings:*

$$Mf \otimes_R S \simeq M\left(X \to B\,GL_1(R) \to B\,GL_1(S)\right).$$

Proof. The map $B\,GL_1(R) \to B\,GL_1(S)$ corresponds to restriction of the functor $(-) \otimes_R S\colon R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$ to the full subgroupoid of R-modules equivalent to R. The lemma then follows from the definition of Mf as a colimit and the fact that tensoring with S commutes with colimits. □

Example 2.6. Assume $p > 2$. Let $\zeta = \zeta_{p^n-1}$ be a $(p^n - 1)$-th root of unity. Consider the map $f_\zeta\colon \Omega^2 S^3 \to B\,GL_1(\mathbb{S}_p^\wedge[\zeta])$ defined by the element $1 - p \in \pi_1 B\,GL_1(\mathbb{S}_p^\wedge[\zeta]) = (\mathbb{Z}_p[\zeta])^\times$. Then

$$Mf_\zeta \simeq H\mathbb{F}_p[\zeta] \simeq H\mathbb{F}_{p^n}.$$

Example 2.7.

$$H\mathbb{F}_p[x] \simeq M(\Omega^2 S^3 \xrightarrow{f_p} B\,GL_1(\widehat{\mathbb{S}_p}) \to B\,GL_1(\widehat{\mathbb{S}_p}[x])).$$

Example 2.8. Consider the homotopy pushout diagram of $\mathbb{E}_\infty$-rings:

$$\begin{tikzcd} \widehat{\mathbb{S}_p}[x] \arrow[r, "x\mapsto 0"] \arrow[d, "x\mapsto y^2"'] & \widehat{\mathbb{S}_p} \arrow[d] \\ \widehat{\mathbb{S}_p}[y] \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & R \end{tikzcd}$$

After tensoring this diagram with $H\mathbb{Z}_p$, we obtain the homotopy pushout diagram

$$\begin{tikzcd} H\mathbb{Z}_p[x] \arrow[r, "x\mapsto 0"] \arrow[d, "x\mapsto y^2"'] & H\mathbb{Z}_p \arrow[d] \\ H\mathbb{Z}_p[y] \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & H\mathbb{Z}_p \otimes R \end{tikzcd}$$

But because the map $H\mathbb{Z}_p[x] \to H\mathbb{Z}_p[y], x \mapsto y^2$ is a flat map, we can compute the homotopy pushout as usual in $\mathcal{D}(\mathbb{Z}_p)$, where it is isomorphic to $H\mathbb{Z}_p[x]/x^2$. Hence, we have

$$R \otimes H\mathbb{Z}_p \simeq H\mathbb{Z}_p[x]/x^2.$$

Therefore, since $H\mathbb{Z}_p$ is already known to be a Thom spectrum, we can then see that $H\mathbb{Z}_p[x]/x^2$ is a Thom spectrum, via the composite

$$\tau_{\geq 2}\Omega^2 S^3 \to \Omega^2 S^3 \xrightarrow{f_p} B\,GL_1(\widehat{\mathbb{S}_p}) \to B\,GL_1(R),$$

where R is the pushout in the diagram above.

Proposition 2.9. *For a fixed ring spectrum* R*, the Thom spectrum construction defines a functor* $M\colon \mathcal{S}_{/B\,GL_1(R)} \to R\text{-}\mathbf{Mod}$ *that is both colimit-preserving and symmetric monoidal with respect to the Cartesian monoidal structure on its domain.*

3 THH of these Thom spectra

Definition 3.1. Let $(\mathcal{C}, \otimes, I, c)$ be a symmetric monoidal category with symmetry c, and let (A, μ, η) be a monoid in $\mathcal{C}$. The **cyclic bar construction** is the simplicial object $B^{cyc}(A)$ in $\mathcal{C}$ given by $[n] \mapsto A^{\otimes(n+1)}$ with face and degeneracy maps:

$$d_i = \begin{cases} \mathrm{id}_{A^{\otimes i}} \otimes \mu \otimes \mathrm{id}_{A^{\otimes(n-i+1)}} & (i < n) \\ c_{A^{\otimes(n)},A} \circ \mu \otimes \mathrm{id}_{A^{\otimes(n-1)}} & (i = n) \end{cases}$$

$$s_i = \mathrm{id}_{A^{\otimes(i+1)}} \otimes \eta \otimes \mathrm{id}_{A^{\otimes(n-i)}}$$

If $\mathcal{C}$ is instead a symmetric monoidal ∞-category, we can define a similar object as a functor from the nerve $N(\Delta^{op})$ to $\mathcal{C}$.

Example 3.2. If $\mathcal{C} = k$-**Mod**, then $B^{cyc}(A)$ corresponds to the complex defining Hochschild homology, and its realization is a space $HH(A)$ whose homotopy is Hochschild homology of A.

Definition 3.3. Let A be a ring spectrum. The **topological Hochschild homology** of A is the realization of the cyclic bar construction of A in the category of spectra, i.e.

$$THH(A) := |B^{cyc}(A)| = \operatorname*{colim}_{N\Delta^{op}}([n] \mapsto A^{\otimes n}).$$

Definition 3.4. Let A be an R-module spectrum. The **relative** THH of A is the realization of the cyclic bar construction of A in the category of R-module spectra. It is denoted $THH(A/R)$.

We have already seen how we can use Theorem 1.3 to compute THH in various cases:

$$THH(H\mathbb{F}_p) \simeq H\mathbb{F}_p \otimes \Omega S^3$$
$$THH(H\mathbb{Z}_p) \simeq H\mathbb{Z}_p \otimes \tau_{\geq 3}\Omega S^3$$
$$THH(H\mathbb{Z}) \simeq H\mathbb{Z} \otimes \tau_{\geq 3}\Omega S^3$$

but what about the new Thom spectra? The flaw with Theorem 1.3 is that it computes relative THH, not absolute THH (relative to $\mathbb{S}$). We want to compute $THH(\mathbb{F}_p[x])$, for example, but Theorem 1.3 can only tell us $THH(\mathbb{F}_p[x]/\widehat{\mathbb{S}_p}[x])$.

We can try to address some of these difficulties with the **base change property** of THH: if A and B are R-algebra spectra, then

$$THH(B/R) \otimes_{THH(A/R)} A \simeq THH(B/A).$$

Let's try to use this to compute $THH(\mathbb{F}_p[x])$ from $THH(\mathbb{F}_p[x]/\mathbb{S}[x])$: $R = \mathbb{S}$, $A = \mathbb{S}[x]$, and $B = \mathbb{F}_p[x]$. Then:

$$THH(\mathbb{F}_p[x]) \otimes_{THH(\mathbb{S}[x])} \mathbb{S}[x] \simeq THH(\mathbb{F}_p[x]/\mathbb{S}[x]).$$

The problem now is that we're working backwards, and we need to know $THH(\mathbb{S}[x])$.