Higher Groupoids and Categories

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The goal of our talk today is to give a description of the categories of fibrant objects and show that higher groupoids and higher categories are examples of categories of fibrant objects.

1 Categories of Fibrant Objects

Definition 1.1. A **category with weak equivalences** is a category $\mathcal{V}$ together with a subcategory $\mathcal{W}$ which

- contains all isomorphisms of $\mathcal{V}$;
- satisfies two-out-of-three: for any two composable morphisms f, g of $\mathcal{V}$, if two of $f, g, g \circ f$ are in $\mathcal{W}$, then so is the third.

Definition 1.2. A **category of fibrant objects** is a small category $\mathcal{V}$ with weak equivalences $\mathcal{W}$ and a subcategory $\mathcal{F} \subset \mathcal{V}$ of fibrations, satisfying the following axioms. Here, we refer to morphisms which are both fibrations and weak equivalences as trivial fibrations.

(F1) there is a terminal object e in $\mathcal{V}$, and $(X \to e) \in \mathcal{F}, \forall X \in Ob(\mathcal{V})$, i.e. any morphism with target e in $\mathcal{V}$ is a fibration.

(F2) pullbacks of fibrations are fibrations;

(F3) pullbacks of trivial fibrations are trivial fibrations;

(F4) every morphism $f : X \to Y$ has a factorization

$$\begin{tikzcd} & P \arrow[dr, "p"] & \\ X \arrow[ur, "r", "\sim"'] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

where r is a weak equivalence and p is a fibration.

An object X is called **fibrant** if the morphism $X \to e$ is a fibration. Axiom (F1) states that every object in $\mathcal{V}$ is fibrant.

Associated to a small category with WE is its simplicial realization $L(\mathcal{V}, \mathcal{W})$. This is a category enrich in simplicial sets, with the same object as $\mathcal{V}$, which refines the usual localization.

The reason for the importance of categories of fibrant objects is that they allow a simple realization of the simplicial localization $L(\mathcal{V}, \mathcal{W})$. Namely, by a theorem of Cisinski [Proposition 3.23], the simplicial Hom-set $\mathrm{Hom}(X, Y)$ of morphisms from X to Y in the simplicial localization of a category of fibrant objects is the nerve of the category whose objects are the spans

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

where f is a trivial fibration and whose morphisms are commuting diagrams

$$\begin{tikzcd} & P_0 \arrow[dl, "f_0"'] \arrow[dr, "g_0"] \arrow[dd, "h"] & \\ X & & Y \\ & P_1 \arrow[ul, "f_1"] \arrow[ur, "g_1"'] & \end{tikzcd}$$

Lemma 1.1. *The weak equivalences of a category of fibrant objects are determined by the trivial fibrations: a morphism f is a weak equivalence if and only if it factorizes as a composition qs where q is a trivial fibration and s is a section of trivial fibrations.*

Proof. Let Y be an object of $\mathcal{V}$. The diagonal $Y \to Y \times Y$ has a factorization into a weak equivalence followed by a fibration

$$Y \xrightarrow[\sim]{s} PY \overset{\partial_0 \times \partial_1}{\twoheadrightarrow} Y \times Y$$

The object PY is called a path space of Y.

Since Y is fibrant, the two projection $p_i : Y \times Y \to Y$ are fibrations since they are pullbacks of the fibration $Y \twoheadrightarrow e$

$$\begin{tikzcd}
Y \times Y \arrow[r, two heads, "p_0"] \arrow[d, two heads, "p_1"'] & Y \arrow[d, two heads] \\
Y \arrow[r, two heads] & e
\end{tikzcd}$$

and it follows that the morphisms $\partial_i : PY \overset{\partial_0 \times \partial_1}{\twoheadrightarrow} Y \times Y \overset{p_i}{\twoheadrightarrow} Y$ are fibrations. Note they are weak equivalences since the compositions

$$Y \xrightarrow[\sim]{s} PY \overset{\partial_0 \times \partial_1}{\twoheadrightarrow} Y \times Y \overset{p_i}{\twoheadrightarrow} Y$$

are identities, thus weak equivalences, and by 2-of-3, $\partial_i : PY \overset{\sim}{\twoheadrightarrow} Y$ are trivial fibrations.

Given a morphism $f : X \to Y$ and the pullback

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s(f)"', "\wr"] \arrow[dd, bend right=60, "\mathrm{Id}"'] & Y \arrow[d, "s", "\wr"'] \arrow[dd, bend left=60, "\mathrm{Id}"] \\
P(f) \arrow[r, "\pi"] \arrow[d, two heads, "p(f)"', "\wr"] & PY \arrow[d, two heads, "f", "\wr"'] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

we see that the projection $p(f) : P(f) \overset{\sim}{\twoheadrightarrow} X$ is a trivial fibration, with sections $s(f) : X \to P(f)$ induced by $s : Y \to PY$ and $f : X \to Y$. Note $s(f)$ is a weak equivalence by 2-of-3.

We can also express $P(f)$ as a pullback of the upper square

$$\begin{tikzcd}
P(f) \arrow[r, "\pi"] \arrow[d, two heads, "p(f) \times q(f)"'] & PY \arrow[d, two heads, "\partial_0 \times \partial_1"] \\
X \times Y \arrow[r, "f \times 1"] \arrow[d, two heads, "p_0"'] & Y \times Y \arrow[d, two heads, "p_0"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

which shows that $p(f) \times q(f)$ is a fibration. Furthermore, $p_1 : X \times Y \twoheadrightarrow Y$ is a fibration since X is fibrant and p_1 is the pullback of a fibration, so $q(f) : P(f) \twoheadrightarrow X \times Y \overset{p_1}{\twoheadrightarrow} Y$ is the composite of fibrations and thus a fibration.

In this way, we obtain the desired factorization of f:

$$\begin{tikzcd} & P(f) \arrow[dr, "q(f)"] & \\ X \arrow[ur, "s(f)"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

□

Remark 1.1. The proof of this lemma show that axiom (F4) is implied by the following special case:

(F4') each diagonal morphism $\Delta : X \to X \times X$ factorizes as

$$\begin{tikzcd} & PX \arrow[dr, "q"] & \\ X \arrow[ur, "s", "\sim"'] \arrow[rr, "\Delta"'] & & X \times X \end{tikzcd}$$

where s is a weak equivalence and q is a fibration.

2 Descent category

Definition 2.1. (Descent category) A small category $\mathcal{V}$ of **spaces**, together with a subcategory of **covers**, satisfying

(D1) $\mathcal{V}$ has finite limits;

(D2) the pullback of a cover is a cover;

(D3) if f is a cover and gf is a cover, then g is a cover.

is called a **descent category**.

Example 2.1.

1. Topos with epimorphisms as covers;
2. category of schemes with surjective étale morphisms/smooth epimorphisms/faithful flat morphisms as covers;
3. category of Banach analytic spaces with surjective submersions as covers.

Definition 2.2. The **kernel pair** of a morphism $f : X \to Y$ in a category with finite limits is the diagram

$$X \times_Y X \rightrightarrows X$$

The coequalizer p of the kernel pair of f, if exists, is called the **coimage** of f:

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, "p"] \arrow[rr, bend left, "f"] & Z \arrow[r, dotted, "i"] & Y \end{tikzcd}$$

The **image** of f is the morphism $i : Z \to Y$.

Definition 2.3. A morphism $f : X \to Y$ in a category $\mathcal{V}$ is an **effective epimorphism** if p equals f, in the sense that i is an isomorphism. In other words, f is the coequalizer of some kernel pair.

One of the reason for the importance of effective epimorphisms is that pullback along an effective epimorphism is conservative, i.e. reflecting isomorphisms.

Definition 2.4. A descent category is **subcanonical** is covers are effective epimorphisms.

All the categories which we have defined above have this property.

Definition 2.5. A category is **regular** if

- it has finite limits;
- kernel pairs have coequalizers;
- the pullback of an effective epimorphism along any morphism is again an effective epimorphism.

A regular category admits a good notion of image factorization, i.e., every morphism factors into an effective epimorphism followed by a monomorphism, and such a factorization is unique up to isomorphism.

Regular epimorphisms are not generally closed under composition. But in a regular category, this is true.

Proposition 2.1. *In a regular category $\mathcal{C}$, regular epimorphisms and strong epimorphisms (= extremal epimorphisms since $\mathcal{C}$ admits pullbacks) coincide; hence regular epimorphisms are closed under composition.*

Definition 2.6. A regular descent category is a subcanonical descent category together with a subcategory of regular morphisms satisfying the following axioms:

(R1) every cover is regular;

(R2) the pullback of a regular morphism is regular;

(R3) every regular morphism has a coimage, and its coimage is a cover.

Lemma 2.1. *Let $\mathcal{V}$ be a regular descent category, and consider the factorization of a regular morphism $f : X \to Y$ into a cover $p : X \to Z$ followed by a morphism $i : Z \to Y$, then i is a monomorphism.*

Proof. Consider

$$\begin{tikzcd}
X \times_Y X \arrow[r] \arrow[d] & Z \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
X \times_Y Z \arrow[r] \arrow[d] & Z \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d, "i"] \\
Z \arrow[r, "p"] & Z \arrow[r, "i"] & Y
\end{tikzcd}$$

where all the four squares are pullbacks. Then $p \times_Y p : X \times_Y X \to Z \times_Y Z$ is the composition of covers, and thus a cover, so it is an effective epimorphism. Note

$$\pi_1 \circ (p \times_Y p) = \pi_2 \circ (p \times_Y p)\,,$$

so $\pi_1 = \pi_2 : Z \times_Y Z \to Z$, which implies that i is a monomorphism.

$\square$

3 k-groupoids

3.1 Definitions

Definition 3.1. A simplicial object in a descent category is called a s**implicial space**.

A finite simplicial set is a simplicial set having only a finite number of nondegenerate simplicies. Given a simplicial space X and a finite simplicial set T, let $\mathrm{Hom}\,(T, X)$ be the space of simplicial morphisms from T to X. This is a finite limit in $\mathcal{V}$ and its existence is guaranteed by (D1). In particular, By Yoneda embedding, $X_n = \mathrm{Hom}\,(\Delta^n, X)$.

Definition 3.2. Let T be a finite simplicial set and $i : S \hookrightarrow T$ be a simplicial subset. If $f : X \to Y$ is a morphism of simplicial spaces, define

$$\mathrm{Hom}\,(i, f) = \mathrm{Hom}\,(S, X) \times_{\mathrm{Hom}(S,Y)} \mathrm{Hom}\,(T, Y)$$

give by the pullback

$$\begin{tikzcd} \mathrm{Hom}\,(i,f) \arrow[r] \arrow[d] & \mathrm{Hom}\,(T,Y) \arrow[d, "i^*"] \\ \mathrm{Hom}\,(S,X) \arrow[r, "f_*"] & \mathrm{Hom}\,(S,Y) \end{tikzcd}$$

Definition 3.3. Let $n \geq 0$ be a natural number. The **matching space** $\mathrm{Hom}\,(\partial\Delta^n, X)$ of a simplicial space X is the finite limit $\mathrm{Hom}\,(\partial\Delta^n, X)$ which represents the simplicial morphisms from $\partial\Delta^n$ to X. More generally, the matching space of a simplicial morphism $f : X \to Y$ is the finite limit

$$\mathrm{Hom}\,(\partial\Delta^n \hookrightarrow \Delta^n, f) = \mathrm{Hom}\,(\partial\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n . Y)} Y_n.$$

Definition 3.4. A simplicial morphism $f : X \to Y$ in $s\mathcal{V}$ is a **hypercover** if for all $n \geq 0$ the morphism

$$X_n \longrightarrow \mathrm{Hom}\,(\partial\Delta^n \hookrightarrow \Delta^n, f)$$

is a cover.

Lemma 3.1. *Let T be a finite simplicial set and $i : S \hookrightarrow T$ be a simplicial subset. If $f : X \to Y$ is a hypercover, then the induced morphism*

$$Hom\,(T, X) \longrightarrow Hom\,(i, f)$$

is a cover.

Proof. There is a filtration of T

$$S = F_{-1}T \subset F_0 T \subset \cdots \subset T$$

satisfying the following conditions:

1. $T = \bigcup_l F_l T$;

2. there is a weakly monotone sequence $n_l, l \geq 0$ and maps $x_l : \Delta^{n_l} \to F_l T$ and $y_l : \partial\Delta^{n_l} \to F_{l-1}T$ such that the following diagram is a pushout

$$\begin{tikzcd} \partial\Delta^{n_l} \arrow[r, "y_l"] \arrow[d] & F_{l-1}T \arrow[d] \\ \Delta^{n_l} \arrow[r, "x_l"] & F_l T \end{tikzcd}$$

Then the morphism

$$\mathrm{Hom}\,(F_l T, X) \longrightarrow \mathrm{Hom}\,(F_{l-1}T \hookrightarrow F_l T, f)$$

is a cover since it is the pullback of the cover $X_{n_l} \to \mathrm{Hom}\,(\partial\Delta^{n_l} \hookrightarrow \Delta^{n_l}, f)$. □

3.2 Intuition

Example 3.1. Let $\Lambda_i^n \subseteq \Delta^n$ be the horn which consists of the union of all but the i-th face of the n-simplex, a simplicial set X is the nerve of a groupoid precisely when the induced morphism

$$X_n \longrightarrow \mathrm{Hom}\,(\Lambda_i^n, X)$$

is an isomorphism for $n > 1$.

Example 3.2. Let A be a simplicial abelian group, the associated complex of normalized chains vanishes after degree k if and only if the morphism $A_n \to \operatorname{Hom}(\Lambda_i^n, A)$ is an isomorphism for $n > k$.

Motivated by these examples, Duskin defines a k-groupoid to be a simplicial set X such that the morphisms $X_n \longrightarrow \operatorname{Hom}(\Lambda_i^n, X)$ is surjective for $n > 0$ and bijective for $n > k$.

Here, we generalize Duskin's theory of k-groupoids to descent categories.

Definition 3.5. Let $k \in \mathbb{N}$. A simplicial space X in a descent category $\mathcal{V}$ is a k-**groupoid** if for each $0 \le i \le n$, the morphism

$$X_n \longrightarrow \operatorname{Hom}(\Lambda_i^n, X)$$

is a cover for $n > 0$ and an isomorphism for $n > k$.

Denote by $s_k\mathcal{V}$ the category of k-groupoids with morphisms the simplicial morphisms of the underlying simplicial spaces. Thus the category $s_0\mathcal{V}$ of 0-groupoids is equivalent to $\mathcal{V}$, and the category $s_1\mathcal{V}$ of 1-groupoids is equivalent to the category of Lie groupoids in $\mathcal{V}$, i.e., groupoids such that the source and target maps are covers. (The equivalence is induced by mapping a Lie groupoid to its nerve.)

Definition 3.6. A morphism $f : X \to Y$ of k-groupoids is a **fibration** if the morphism

$$X_n \longrightarrow \operatorname{Hom}(\Lambda_i^n \hookrightarrow \Delta^n, f)$$

is a cover for $0 \le i \le n$ and $n > 0$.

3.3 The category of k-groupoids

Theorem 3.1. *The category of k-groupoids $s_k\mathcal{V}$ is a category of fibrant objects with fibrations and hypercovers as fibrations and trivial fibrations.*

The key of the proof is similar to the idea in the proof of Lemma 3.1.

Definition 3.7. Let $m > 0$. An m-**expansion** $i : S \hookrightarrow T$ is a map of simplicial such that there exists a filtration

$$S = F_{-1}T \subset F_0T \subset \cdots \subset T$$

satisfying the following conditions:

1. $T = \bigcup_l F_l T$;

2. there is a weakly monotone sequence $n_l, l \geq 0$ and maps $x_l : \Delta^{n_l} \to F_l T$ and $y_l : \Lambda^{n_l}_{i_l} \to F_{l-1} T$ such that the following diagram is a pushout

$$\begin{tikzcd} \Lambda^{n_l}_{i_l} \arrow[r, "y_l"] \arrow[d] & F_{l-1}T \arrow[d] \\ \Delta^{n_l} \arrow[r, "x_l"] & F_l T \end{tikzcd}$$

Lemma 3.2. *If $S \subset \Delta^n$ is the union of $0 < m \leq n$ faces of Δ^n, then the inclusion $i : S \hookrightarrow \Delta^n$ is an m-expansion.*

Lemma 3.3. *Let T be a finite simplicial set and $i : S \hookrightarrow T$ be an m-expansion.*

1. *If X is a k-groupoid, the induced morphism*

$$Hom\,(T, X) \longrightarrow Hom\,(S, X)$$

   is a cover, and an isomorphism if $m > k$.

2. *If $f : X \to Y$ is a fibration of k-groupoids, the induced morphism*

$$Hom\,(T, X) \longrightarrow Hom\,(i, f)$$

   is a cover, and an isomorphism if $m > k$.

Corollary 3.1. *If X is a k-groupoid, the face map $\partial_i : X_n \to X_{n-1}$ is a cover.*

Lemma 3.4. *If $f : X \to Y$ is a fibration of k-groupoids, then*

$$\alpha : X_n \longrightarrow Hom\,(\Lambda^n_i \hookrightarrow \Delta^n, f)$$

is an isomorphism for $n > k$.

Proof. We have the following commuting diagram in which the square is a pullback

$$\begin{tikzcd}
X_n \arrow[r] \arrow[dr, "\cong", "\beta"'] & \mathrm{Hom}\,(\Lambda^n_i \hookrightarrow \Delta^n, f) \arrow[r] \arrow[d, "\cong"'] & Y_n \arrow[d, "\cong"', "\gamma"] \\
& \mathrm{Hom}\,(\Lambda^n_i, X) \arrow[r] & \mathrm{Hom}\,(\Lambda^n_i, Y)
\end{tikzcd}$$

Since β and γ are isomorphisms, so is α. ☐

Lemma 3.5. *A hypercover $f : X \to Y$ of k-groupoids is a fibration.*

Proof. For $n > 0$ and $0 \leq i \leq n$, we have the following commutative diagram in which the square is a pullback:

$$\begin{tikzcd}
X_n \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathrm{Hom}\,(\partial\Delta^n \hookrightarrow \Delta^n, f) \arrow[r] \arrow[d] & X_{n-1} \arrow[d, "\gamma"] \\
& \mathrm{Hom}\,(\Lambda^n_i \hookrightarrow \Delta^n, f) \arrow[r] & \mathrm{Hom}\left(\partial\Delta^{n-1} \hookrightarrow \Delta^{n-1}, f\right)
\end{tikzcd}$$

note α and γ are covers, so β is a cover. ☐

Lemma 3.6. *Fibrations and hypercovers are closed under composition.*

Lemma 3.7. *If $p : X \to Y$ is a hypercover and $f : Z \to Y$ is a morphism, then the pullback q of p along f is a hypercover.*

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d, "q"'] & X \arrow[d, "p"] \\
Z \arrow[r, "f"] & Y
\end{tikzcd}$$

Lemma 3.8. *If $p : X \to Y$ is a fibration of k-groupoids and $f : Z \to Y$ is a morphism of k-groupoids, then $X \times_Y Z$ is a k-groupoid and the morphism q in the pullback diagram*

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d, "q"'] & X \arrow[d, "p"] \\
Z \arrow[r, "f"] & Y
\end{tikzcd}$$

is a fibration.

Next, we show that $s\mathcal{V}$ is a descent category with hypercovers as covers.

Lemma 3.9. *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of simplicial spaces and f, gf are hypercovers, then g is a hypercover.*

In order to show that k-groupoids form a category of fibrant objects, we need to construct path spaces.

Definition 3.8. Let $P_n : s\mathcal{V} \to s\mathcal{V}$ be the functor on simplicial spaces such that

$$(P_n X)_m = \mathrm{Hom}\left(\Delta^{m,n}, X\right),$$

where $\Delta^{m,n} = \Delta^m \times \Delta^n$.

The functor P_n is the space of maps from Δ^n to X. In particular, $P_0 X = X$ and $PX = P_1 X$ is a path space of X.

Definition 3.9. A morphism $f : X \to Y$ of k-groupoids is a **weak equivalence** if the fibration

$$q(f) : P(f) \to Y$$

is a hypercover, where $P(f) = X \times_Y P_1 Y$.

Remark 3.1. In case of Kan complexes, this characterization of weak equivalences amounts to the vanishing of the relative simplicial homotopy groups.

Lemma 3.10. *The simplicial morphism $P_1 X \to X \times X$ is a fibration and the face maps $P_1 X \to X$ are hypercovers. So the factorization axiom holds in $s_k\mathcal{V}$.*

Lemma 3.11. *The weak equivalences form a subcategory of $s_k\mathcal{V}$.*

Lemma 3.12. *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of k-groupoids such that f and gf are weak equivalences, then g is a weak equivalence.*

Lemma 3.13. *A fibration $f : X \to Y$ is a weak equivalence if and only if it is a hypercover.*

Proof. In the following commutative diagram the solid arrows are hypercovers

$$\begin{tikzcd} P_1 X \arrow[r] \arrow[d] & P(f) \arrow[d, dashed, "q(f)"] \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd}$$

and lemma follows from lemma 3.9. □

Lemma 3.14. *If $f : X \to Y$ is a fibration and $g : Y \to Z$ are gf are hypercovers, then f is a hypercover.*

Lemma 3.15. *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of k-groupoids such that g, gf are weak equivalences, then f is a weak equivalence.*

The following theorem is analogous to Gabriel and Zisman's famous theorem on anodyne extensions.

Theorem 3.2. *A morphism $f : X \to Y$ is a weak equivalence if and only if the morphisms*

$$Hom\left(\Delta^n \hookrightarrow \Delta^{n+1}, f\right) \longrightarrow Hom\left(\partial\Delta^{n+1} \hookrightarrow \Lambda^{n+1}_{n+1}, f\right)$$

are covers for $n \geq 0$.

4 k-categories

Recall the thick 1-simplex $\mathbf{\Delta}^1$ is the nerve of the groupoid $[[1]]$ with objects $\{0,1\}$ and a single morphism between any pair of objects.

Let $k > 0$.

Definition 4.1. A **k-category** in a descent category $\mathcal{V}$ is a simplicial space X such that

1. if $0 < i < n$, the morphisms

$$X_n \longrightarrow \mathrm{Hom}\left(\Lambda_i^n, X\right)$$

   is a cover and an isomorphism if $n > k$.

2. if $i = 0, 1$, the morphism

$$\mathrm{Hom}\left(\mathbf{\Delta}^1, X\right) \longrightarrow \mathrm{Hom}\left(\Lambda_i^1, X\right) \cong X_0$$

   is a cover.

Lemma 4.1. *A k-category X is $(k+1)$-coskeletal, that is, for any $n \geq 0$,*

$$X_n \cong cosk_{k+1} X_n = Hom\left(sk_{k+1}\Delta^n, X\right).$$

Proposition 4.1. *A k-groupoid is a k-category.*

Definition 4.2. A **quasi-fibration** $f : X \to Y$ of k-categories is a morphism of the underlying simplicial spaces such that

1. if $0 < i < n$, the morphisms

$$X_n \longrightarrow \mathrm{Hom}\left(\Lambda_i^n \hookrightarrow \Delta^n, f\right)$$

   is a cover;

2. if $i = 0, 1$, the morphism

$$\mathrm{Hom}\left(\mathbf{\Delta}^1, X\right) \longrightarrow \mathrm{Hom}\left(\Delta^0 \hookrightarrow \mathbf{\Delta}^1, f\right) = X_0 \times_{Y^0} \mathrm{Hom}\left(\mathbf{\Delta}^1, Y\right)$$

is a cover.

Theorem 4.1. *We have a functor $X \mapsto (X)$ from k-categories to k-groupoids which may be interpreted as the k-groupoid of quasi-invertible morphisms in X.*

1. *If X is a k-category, then the simplicial space*

$$(X)_n = Hom\left(\mathbf{\Delta}^n, X\right)$$

   is a k-groupoid.

2. *If $f : X \to Y$ is a quasi-fibration of k-categories, then*

$$(f) : (X) \to (Y)$$

   is a fibration of k-groupoids.

3. *If $f : X \to Y$ is a hypercover of k-categories, then*

$$(f) : (X) \to (Y)$$

   is a hypercover of k-groupoids.

Theorem 4.2. *The category of k-categories is a category of fibrant objects.*

5 Example: Nerve of DGA

Let A be a differential graded algebra over a field K with $d : A^{\bullet} \to A^{\bullet+1}$, which is finite-dimensional in each degree and concentrated in degrees $> -k$.

The curvature map is the quadratic polynomial

$$\Phi(\mu) = d\mu + \mu^2 : A^1 \to A^2$$

The Maurer-Cartan locus

$$\mathbf{MC}(A) = V(\Phi) \subset A^1$$

is the zero locus of Φ.

Given $\mu, \nu \in \mathbf{MC}(A)$, define a differential $d_{\mu,\nu}$ on the graded vector space underlying A by

$$a \in A^i \mapsto d_{\mu,\nu} a = da + \mu a - (-1)^i a\nu \in A^{i+1}.$$

Let $C^{\bullet}(\Delta^n)$ be the differential graded algebra of normalized simplicial cochains on Δ^n with coefficients in K.

An element $a \in C^{\bullet}(\Delta^n) \otimes A^{\bullet}$ corresponds to a collection of elements

$$\left(a_{i_0 \cdots i_k} \in A^{i-k} | 0 \le i_0 < \cdots < i_k \le n\right)$$

where $a_{i_0 \cdots i_k}$ is the evaluation of the cochain a on the face of the simplex Δ^n with vertices $\{i_0, \cdots, i_k\}$. The differential on the differential graded algebra $C^{\bullet}(\Delta^n) \otimes A$ is the sum of the simplicial differential on $C^{\bullet}(\Delta^n) \otimes A$ and the internal differential of A:

$$(\delta a)_{i_0 \cdots i_k} = \sum_{l=0}^{k} (-1)^l a_{i_0 \cdots \hat{i}_l \cdots i_k} + (-1)^k d(a_{i_0 \cdots i_k})$$

The product of $C^{\bullet}(\Delta^n) \otimes A$ combines the Alexander-Whitney product on simplicial cochains with the product on A: if a has total degree j, then

$$(a \cup b)_{i_0 \cdots i_k} = \sum_{l=0}^{k} (-1)^{(j-l)(k-l)} a_{i_0 \cdots i_l} b_{i_l \cdots i_k}.$$

The nerve of a differential graded algebra A is the simplicial scheme $N_\bullet A$ such that $N_n A$ is the MaurerCartan locus of $C^\bullet(\Delta^n) \otimes A$:

$$N_n A = \mathbf{MC}\left(C^\bullet(\Delta^n) \otimes A\right)$$

Theorem 5.1. *Let A be a differential graded algebra such that A^i is finite-dimensional for $i \leq 1$, and vanishes for $i \leq -k$. Then $N_\bullet A$ is a (regular) k-category.*

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