Higher Groupoids and Categories

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The goal of our talk today is to give a description of the categories of fibrant objects and show that higher groupoids and higher categories are examples of categories of fibrant objects.

1 Categories of Fibrant Objects

Definition 1.1. A category with weak equivalences is a category $\mathcal{V}$ together with a subcategory $\mathcal{W}$ which

- contains all isomorphisms of $\mathcal{V}$;

- satisfies two-out-of-three: for any two composable morphisms $f, g$ of $\mathcal{V}$, if two of $f, g, g \circ f$ are in $\mathcal{W}$, then so is the third.

Definition 1.2. A category of fibrant objects is a small category $\mathcal{V}$ with weak equivalences $\mathcal{W}$ and a subcategory $\mathcal{F} \subset \mathcal{V}$ of fibrations, satisfying the following axioms. Here, we refer to morphisms which are both fibrations and weak equivalences as trivial fibrations.

(F1) there is a terminal object $e$ in $\mathcal{V}$, and $(X \to e) \in \mathcal{F}, \forall X \in Ob(\mathcal{V})$, i.e. any morphism with target $e$ in $\mathcal{V}$ is a fibration.

(F2) pullbacks of fibrations are fibrations;

(F3) pullbacks of trivial fibrations are trivial fibrations;
(F4) every morphism $f : X \rightarrow Y$ has a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{r} & & \downarrow{p} \\
X & \sim & Y
\end{array}
$$

where $r$ is a weak equivalence and $p$ is a fibration.

An object $X$ is called **fibrant** if the morphism $X \rightarrow e$ is a fibration. Axiom (F1) states that every object in $\mathcal{V}$ is fibrant.

Associated to a small category with WE is its simplicial realization $L(\mathcal{V}, \mathcal{W})$. This is a category enriched in simplicial sets, with the same object as $\mathcal{V}$, which refines the usual localization.

The reason for the importance of categories of fibrant objects is that they allow a simple realization of the simplicial localization $L(\mathcal{V}, \mathcal{W})$. Namely, by a theorem of Cisinski [Proposition 3.23], the simplicial Hom-set $\text{Hom}(X, Y)$ of morphisms from $X$ to $Y$ in the simplicial localization of a category of fibrant objects is the nerve of the category whose objects are the spans

$$
\begin{array}{ccc}
P & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
X & \xleftarrow{f_0} & P_0
\end{array}
$$

where $f$ is a trivial fibration and whose morphisms are commuting diagrams

$$
\begin{array}{ccc}
P_0 & \xrightarrow{g_0} & Y \\
\downarrow{h} & & \downarrow{g_1} \\
P_1 & \xleftarrow{f_1} & X
\end{array}
$$

**Lemma 1.1.** The weak equivalences of a category of fibrant objects are determined by the trivial fibrations: a morphism $f$ is a weak equivalence if and only if it factorizes as a composition $qs$ where $q$ is a trivial fibration and $s$ is a section of trivial fibrations.

**Proof.** Let $Y$ be an object of $\mathcal{V}$. The diagonal $Y \rightarrow Y \times Y$ has a factorization into a weak equivalence
followed by a fibration

\[ Y \xrightarrow{s} PY \xrightarrow{\partial_0 \times \partial_1} Y \times Y \]

The object \( PY \) is called a path space of \( Y \).

Since \( Y \) is fibrant, the two projection \( p_i : Y \times Y \to Y \) are fibrations since they are pullbacks of the fibration \( Y \to e \)

\[
\begin{array}{ccc}
Y \times Y & \xrightarrow{p_0} & Y \\
\downarrow & & \downarrow \\
Y & \to & e
\end{array}
\]

and it follows that the morphisms \( \partial_i : PY \to Y \times Y \xrightarrow{p_i} Y \) are fibrations. Note they are weak equivalences since the compositions

\[
\begin{array}{ccc}
Y & \xrightarrow{s} & PY \\
\downarrow & & \downarrow \\
Y \times Y & \xrightarrow{p_i} & Y
\end{array}
\]

are identities, thus weak equivalences, and by 2-of-3, \( \partial_i : PY \to Y \) are trivial fibrations.

Given a morphism \( f : X \to Y \) and the pullback

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
P(f) & \xrightarrow{\pi} & PY
\end{array}
\]

we see that the projection \( p(f) : P(f) \to X \) is a trivial fibration, with sections \( s(f) : X \to P(f) \) induced by \( s : Y \to PY \) and \( f : X \to Y \). Note \( s(f) \) is a weak equivalence by 2-of-3.

We can also express \( P(f) \) as a pullback of the upper square

\[
\begin{array}{ccc}
P(f) & \xrightarrow{\pi} & PY \\
\downarrow & \downarrow & \downarrow \\
X \times Y & \xrightarrow{f \times 1} & Y \times Y \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]
which shows that \( p(f) \times q(f) \) is a fibration. Furthermore, \( p_1 : X \times Y \to Y \) is a fibration since \( X \) is fibrant and \( p_1 \) is the pullback of a fibration, so \( q(f) : P(f) \to X \times Y \xrightarrow{p_1} Y \) is the composite of fibrations and thus a fibration.

In this way, we obtain the desired factorization of \( f \):

![Diagram](image)

\[ \square \]

Remark 1.1. The proof of this lemma show that axiom (F4) is implied by the following special case:

(F4') each diagonal morphism \( \Delta : X \to X \times X \) factorizes as

![Diagram](image)

where \( s \) is a weak equivalence and \( q \) is a fibration.
2 Descent category

Definition 2.1. (Descent category) A small category $V$ of spaces, together with a subcategory of covers, satisfying

(D1) $V$ has finite limits;

(D2) the pullback of a cover is a cover;

(D3) if $f$ is a cover and $gf$ is a cover, then $g$ is a cover.

is called a descent category.

Example 2.1.

1. Topos with epimorphisms as covers;

2. category of schemes with surjective étale morphisms/smooth epimorphisms/faithful flat morphisms as covers;

3. category of Banach analytic spaces with surjective submersions as covers.

Definition 2.2. The kernel pair of a morphism $f : X \to Y$ in a category with finite limits is the diagram

$$
\begin{array}{ccc}
X \times_Y X & \longrightarrow & X \\
\downarrow & & \downarrow \\
X & \longrightarrow & X
\end{array}
$$

The coequalizer $p$ of the kernel pair of $f$, if exists, is called the coimage of $f$:

$$
\begin{array}{ccc}
X \times_Y X & \longrightarrow & X & \longrightarrow & Z & \longrightarrow & Y \\
\downarrow & \Downarrow{f} & & & & & \\
X & \longrightarrow & X & \longrightarrow & Y
\end{array}
$$

The image of $f$ is the morphism $i : Z \to Y$.

Definition 2.3. A morphism $f : X \to Y$ in a category $V$ is an effective epimorphism if $p$ equals $f$, in the sense that $i$ is an isomorphism. In other words, $f$ is the coequalizer of some kernel pair.

One of the reason for the importance of effective epimorphisms is that pullback along an effective epimorphism is conservative, i.e. reflecting isomorphisms.
Definition 2.4. A descent category is subcanonical if covers are effective epimorphisms.

All the categories which we have defined above have this property.

Definition 2.5. A category is regular if

- it has finite limits;
- kernel pairs have coequalizers;
- the pullback of an effective epimorphism along any morphism is again an effective epimorphism.

A regular category admits a good notion of image factorization, i.e., every morphism factors into an effective epimorphism followed by a monomorphism, and such a factorization is unique up to isomorphism.

Regular epimorphisms are not generally closed under composition. But in a regular category, this is true.

Proposition 2.1. In a regular category C, regular epimorphisms and strong epimorphisms (= extremal epimorphisms since C admits pullbacks) coincide; hence regular epimorphisms are closed under composition.

Definition 2.6. A regular descent category is a subcanonical descent category together with a subcategory of regular morphisms satisfying the following axioms:

(R1) every cover is regular;

(R2) the pullback of a regular morphism is regular;

(R3) every regular morphism has a coimage, and its coimage is a cover.

Lemma 2.1. Let V be a regular descent category, and consider the factorization of a regular morphism $f : X \to Y$ into a cover $p : X \to Z$ followed by a morphism $i : Z \to Y$, then $i$ is a monomorphism.
Proof. Consider

\[
\begin{array}{cccccc}
X \times_Y X & \rightarrow & Z \times_Y X & \rightarrow & X \\
\downarrow & & \downarrow & & \downarrow^p \\
X \times_Y Z & \rightarrow & Z \times_Y Z & \rightarrow & Z \\
\downarrow & & \downarrow & & \downarrow^i \\
Z & \rightarrow & Z & \rightarrow & Y \\
\end{array}
\]

where all the four squares are pullbacks. Then \( p \times_Y p : X \times_Y X \rightarrow Z \times_Y Z \) is the composition of covers, and thus a cover, so it is an effective epimorphism. Note

\[
\pi_1 \circ (p \times_Y p) = \pi_2 \circ (p \times_Y p),
\]

so \( \pi_1 = \pi_2 : Z \times_Y Z \rightarrow Z \), which implies that \( i \) is a monomorphism.
3 \( k \)-groupoids

3.1 Definitions

Definition 3.1. A simplicial object in a descent category is called a simplicial space.

A finite simplicial set is a simplicial set having only a finite number of nondegenerate simplicies. Given a simplicial space \( X \) and a finite simplicial set \( T \), let \( \text{Hom}(T, X) \) be the space of simplicial morphisms from \( T \) to \( X \). This is a finite limit in \( \mathcal{V} \) and its existence is guaranteed by (D1). In particular, By Yoneda embedding, \( X_n = \text{Hom}(\Delta^n, X) \).

Definition 3.2. Let \( T \) be a finite simplicial set and \( i : S \to T \) be a simplicial subset. If \( f : X \to Y \) is a morphism of simplicial spaces, define

\[
\text{Hom}(i, f) = \text{Hom}(S, X) \times_{\text{Hom}(S,Y) \text{Hom}(T,Y)} \text{Hom}(T, Y)
\]

give by the pullback

\[
\begin{array}{ccc}
\text{Hom}(i, f) & \longrightarrow & \text{Hom}(T, Y) \\
\downarrow & & \downarrow i^* \\
\text{Hom}(S, X) & \xrightarrow{f_*} & \text{Hom}(S, Y)
\end{array}
\]

Definition 3.3. Let \( n \geq 0 \) be a natural number. The matching space \( \text{Hom}(\partial \Delta^n, X) \) of a simplicial space \( X \) is the finite limit \( \text{Hom}(\partial \Delta^n, X) \) which represents the simplicial morphisms from \( \partial \Delta^n \) to \( X \). More generally, the matching space of a simplicial morphism \( f : X \to Y \) is the finite limit

\[
\text{Hom}(\partial \Delta^n \to \Delta^n, f) = \text{Hom}(\partial \Delta^n, X) \times_{\text{Hom}(\partial \Delta^n, Y)} Y_n.
\]

Definition 3.4. A simplicial morphism \( f : X \to Y \) in \( s\mathcal{V} \) is a hypercover if for all \( n \geq 0 \) the morphism

\[
X_n \longrightarrow \text{Hom}(\partial \Delta^n \to \Delta^n, f)
\]

is a cover.
Lemma 3.1. Let $T$ be a finite simplicial set and $i : S \hookrightarrow T$ be a simplicial subset. If $f : X \to Y$ is a hypercover, then the induced morphism

$$\text{Hom}(T, X) \to \text{Hom}(i, f)$$

is a cover.

Proof. There is a filtration of $T$

$$S = F_{-1}T \subset F_0T \subset \cdots \subset T$$

satisfying the following conditions:

1. $T = \bigcup_l F_lT$;

2. there is a weakly monotone sequence $n_l, l \geq 0$ and maps $x_l : \Delta^{n_l} \to F_lT$ and $y_l : \partial \Delta^{n_l} \to F_{l-1}T$ such that the following diagram is a pushout

$$
\begin{array}{ccc}
\partial \Delta^{n_l} & \xrightarrow{y_l} & F_{l-1}T \\
\downarrow & & \downarrow \\
\Delta^{n_l} & \xrightarrow{x_l} & F_lT \\
\end{array}
$$

Then the morphism

$$\text{Hom}(F_lT, X) \to \text{Hom}(F_{l-1}T \hookrightarrow F_lT, f)$$

is a cover since it is the pullback of the cover $X_{n_l} \to \text{Hom}(\partial \Delta^{n_l} \hookrightarrow \Delta^{n_l}, f)$. \qed

3.2 Intuition

Example 3.1. Let $\Lambda^n_i \subseteq \Delta^n$ be the horn which consists of the union of all but the $i$-th face of the $n$-simplex, a simplicial set $X$ is the nerve of a groupoid precisely when the induced morphism

$$X_n \to \text{Hom}(\Lambda^n_i, X)$$

is an isomorphism for $n > 1$. 

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Example 3.2. Let $A$ be a simplicial abelian group, the associated complex of normalized chains vanishes after degree $k$ if and only if the morphism $A_n \to \text{Hom}(\Lambda^n_i, A)$ is an isomorphism for $n > k$.

Motivated by these examples, Duskin defines a $k$-groupoid to be a simplicial set $X$ such that the morphisms $X_n \to \text{Hom}(\Lambda^n_i, X)$ is surjective for $n > 0$ and bijective for $n > k$.

Here, we generalize Duskin’s theory of $k$-groupoids to descent categories.

Definition 3.5. Let $k \in \mathbb{N}$. A simplicial space $X$ in a descent category $\mathcal{V}$ is a $k$-groupoid if for each $0 \leq i \leq n$, the morphism

$$X_n \to \text{Hom}(\Lambda^n_i, X)$$

is a cover for $n > 0$ and an isomorphism for $n > k$.

Denote by $s_k \mathcal{V}$ the category of $k$-groupoids with morphisms the simplicial morphisms of the underlying simplicial spaces. Thus the category $s_0 \mathcal{V}$ of 0-groupoids is equivalent to $\mathcal{V}$, and the category $s_1 \mathcal{V}$ of 1-groupoids is equivalent to the category of Lie groupoids in $\mathcal{V}$, i.e., groupoids such that the source and target maps are covers. (The equivalence is induced by mapping a Lie groupoid to its nerve.)

Definition 3.6. A morphism $f : X \to Y$ of $k$-groupoids is a fibration if the morphism

$$X_n \to \text{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, f)$$

is a cover for $0 \leq i \leq n$ and $n > 0$.

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Theorem 3.1. The category of $k$-groupoids $s_k \mathcal{V}$ is a category of fibrant objects with fibrations and hypercovers as fibrations and trivial fibrations.

The key of the proof is similar to the idea in the proof of Lemma 3.1.

Definition 3.7. Let $m > 0$. An $m$-expansion $i : S \hookrightarrow T$ is a map of simplicial such that there exists a filtration

$$S = F_{-1}T \subset F_0T \subset \cdots \subset T$$

satisfying the following conditions:
1. \( T = \bigcup_l F_l T; \)

2. there is a weakly monotone sequence \( n_l, l \geq 0 \) and maps \( x_l : \Delta^{n_l} \to F_l T \) and \( y_l : \Lambda^{n_l}_l \to F_{l-1} T \)

such that the following diagram is a pushout

\[
\begin{array}{ccc}
\Lambda^{n_l}_l & \xrightarrow{y_l} & F_{l-1} T \\
\downarrow & & \downarrow \\
\Delta^{n_l} & \xrightarrow{x_l} & F_l T
\end{array}
\]

**Lemma 3.2.** If \( S \subset \Delta^n \) is the union of \( 0 < m \leq n \) faces of \( \Delta^n \), then the inclusion \( i : S \hookrightarrow \Delta^n \) is an \( m \)-expansion.

**Lemma 3.3.** Let \( T \) be a finite simplicial set and \( i : S \hookrightarrow T \) be an \( m \)-expansion.

1. If \( X \) is a \( k \)-groupoid, the induced morphism

\[
\text{Hom}(T, X) \longrightarrow \text{Hom}(S, X)
\]

is a cover, and an isomorphism if \( m > k \).

2. If \( f : X \to Y \) is a fibration of \( k \)-groupoids, the induced morphism

\[
\text{Hom}(T, X) \longrightarrow \text{Hom}(i, f)
\]

is a cover, and an isomorphism if \( m > k \).

**Corollary 3.1.** If \( X \) is a \( k \)-groupoid, the face map \( \partial_i : X_n \to X_{n-1} \) is a cover.

**Lemma 3.4.** If \( f : X \to Y \) is a fibration of \( k \)-groupoids, then

\[
\alpha : X_n \longrightarrow \text{Hom}(\Lambda^n_i \hookrightarrow \Delta^n, f)
\]

is an isomorphism for \( n > k \).
Proof. We have the following commuting diagram in which the square is a pullback

\[
\begin{array}{ccc}
X_n & \xrightarrow{\beta} & \xrightarrow{\gamma} \Hom(\Lambda^n_i, X) \\
& \xrightarrow{\cong} & \xrightarrow{\cong} \\
\Hom(\Lambda^n_i \hookrightarrow \Delta^n, f) & \xrightarrow{\cong} & \Hom(\Lambda^n_i \hookrightarrow \Delta^n, Y)
\end{array}
\]

Since \( \beta \) and \( \gamma \) are isomorphisms, so is \( \alpha \).

\[ \square \]

Lemma 3.5. A hypercover \( f : X \to Y \) of \( k \)-groupoids is a fibration.

Proof. For \( n > 0 \) and \( 0 \leq i \leq n \), we have the following commutative diagram in which the square is a pullback:

\[
\begin{array}{ccc}
X_n & \xrightarrow{\beta} & \xrightarrow{\gamma} \Hom(\partial \Delta^n \hookrightarrow \Delta^n, f) \\
& \xrightarrow{\cong} & \xrightarrow{\cong} \\
\Hom(\Lambda^i_n \hookrightarrow \Delta^n, f) & \xrightarrow{\cong} & \Hom(\partial \Delta^{n-1} \hookrightarrow \Delta^{n-1}, f)
\end{array}
\]

note \( \alpha \) and \( \gamma \) are covers, so \( \beta \) is a cover.

\[ \square \]

Lemma 3.6. Fibrations and hypercovers are closed under composition.

Lemma 3.7. If \( p : X \to Y \) is a hypercover and \( f : Z \to Y \) is a morphism, then the pullback \( q \) of \( p \) along \( f \) is a hypercover.

\[
\begin{array}{ccc}
X \times_Y Z & \longrightarrow & X \\
\downarrow^q & & \downarrow^p \\
Z & \longrightarrow & Y
\end{array}
\]

Lemma 3.8. If \( p : X \to Y \) is a fibration of \( k \)-groupoids and \( f : Z \to Y \) is a morphism of \( k \)-groupoids, then \( X \times_Y Z \) is a \( k \)-groupoid and the morphism \( q \) in the pullback diagram

\[
\begin{array}{ccc}
X \times_Y Z & \longrightarrow & X \\
\downarrow^q & & \downarrow^p \\
Z & \longrightarrow & Y
\end{array}
\]

is a fibration.

Next, we show that \( \mathcal{V} \) is a descent category with hypercovers as covers.
Lemma 3.9. If $f : X \to Y$ and $g : Y \to Z$ are morphisms of simplicial spaces and $f, gf$ are hypercovers, then $g$ is a hypercover.

In order to show that $k$-groupoids form a category of fibrant objects, we need to construct path spaces.

Definition 3.8. Let $P_n : sV \to sV$ be the functor on simplicial spaces such that

$$(P_nX)_m = \text{Hom}(\Delta^{m,n}, X),$$

where $\Delta^{m,n} = \Delta^m \times \Delta^n$.

The functor $P_n$ is the space of maps from $\Delta^n$ to $X$. In particular, $P_0X = X$ and $PX = P_1X$ is a path space of $X$.

Definition 3.9. A morphism $f : X \to Y$ of $k$-groupoids is a weak equivalence if the fibration

$$q(f) : P(f) \to Y$$

is a hypercover, where $P(f) = X \times_Y P_1Y$.

Remark 3.1. In case of Kan complexes, this characterization of weak equivalences amounts to the vanishing of the relative simplicial homotopy groups.

Lemma 3.10. The simplicial morphism $P_1X \to X \times X$ is a fibration and the face maps $P_1X \to X$ are hypercovers. So the factorization axiom holds in $s_kV$.

Lemma 3.11. The weak equivalences form a subcategory of $s_kV$.

Lemma 3.12. If $f : X \to Y$ and $g : Y \to Z$ are morphisms of $k$-groupoids such that $f$ and $gf$ are weak equivalences, then $g$ is a weak equivalence.

Lemma 3.13. A fibration $f : X \to Y$ is a weak equivalence if and only if it is a hypercover.

Proof. In the following commutative diagram the solid arrows are hypercovers
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and lemma follows from lemma 3.9.

**Lemma 3.14.** If $f : X \to Y$ is a fibration and $g : Y \to Z$ are hypercovers, then $f$ is a hypercover.

**Lemma 3.15.** If $f : X \to Y$ and $g : Y \to Z$ are morphisms of $k$-groupoids such that $g, gf$ are weak equivalences, then $f$ is a weak equivalence.

The following theorem is analogous to Gabriel and Zisman’s famous theorem on anodyne extensions.

**Theorem 3.2.** A morphism $f : X \to Y$ is a weak equivalence if and only if the morphisms

$$\text{Hom}(\Delta^n \hookrightarrow \Delta^{n+1}, f) \longrightarrow \text{Hom}(\partial \Delta^{n+1} \hookrightarrow \Lambda_{n+1}^{n+1}, f)$$

are covers for $n \geq 0$. 
4 \( k \)-categories

Recall the thick 1-simplex \( \Delta^1 \) is the nerve of the groupoid \([1]\) with objects \(\{0,1\}\) and a single morphism between any pair of objects.

Let \( k > 0 \).

**Definition 4.1.** A \( k \)-category in a descent category \( V \) is a simplicial space \( X \) such that

1. if \( 0 < i < n \), the morphisms
   \[
   X_n \longrightarrow \text{Hom} (\Lambda^n_i, X)
   \]
   is a cover and an isomorphism if \( n > k \).

2. if \( i = 0, 1 \), the morphism
   \[
   \text{Hom} (\Delta^1, X) \longrightarrow \text{Hom} (\Lambda_1^1, X) \cong X_0
   \]
   is a cover.

**Lemma 4.1.** A \( k \)-category \( X \) is \((k+1)\)-coskeletal, that is, for any \( n \geq 0 \),

\[
X_n \cong \text{cosk}_{k+1} X_n = \text{Hom} (\text{sk}_{k+1} \Delta^n, X).
\]

**Proposition 4.1.** A \( k \)-groupoid is a \( k \)-category.

**Definition 4.2.** A quasi-fibration \( f : X \rightarrow Y \) of \( k \)-categories is a morphism of the underlying simplicial spaces such that

1. if \( 0 < i < n \), the morphisms
   \[
   X_n \longrightarrow \text{Hom} (\Lambda^n_i \hookrightarrow \Delta^n, f)
   \]
   is a cover;

2. if \( i = 0, 1 \), the morphism
   \[
   \text{Hom} (\Delta^1, X) \longrightarrow \text{Hom} (\Delta^0 \hookrightarrow \Delta^1, f) = X_0 \times_{Y_0} \text{Hom} (\Delta^1, Y)
   \]
Theorem 4.1. We have a functor $X \mapsto (X)$ from $k$-categories to $k$-groupoids which may be interpreted as the $k$-groupoid of quasi-invertible morphisms in $X$.

1. If $X$ is a $k$-category, then the simplicial space

$$(X)_n = \text{Hom}(\Delta^n, X)$$

is a $k$-groupoid.

2. If $f : X \to Y$ is a quasi-fibration of $k$-categories, then

$$(f) : (X) \to (Y)$$

is a fibration of $k$-groupoids.

3. If $f : X \to Y$ is a hypercover of $k$-categories, then

$$(f) : (X) \to (Y)$$

is a hypercover of $k$-groupoids.

Theorem 4.2. The category of $k$-categories is a category of fibrant objects.
5 Example: Nerve of DGA

Let $A$ be a differential graded algebra over a field $K$ with $d : A^\bullet \to A^{\bullet+1}$, which is finite-dimensional in each degree and concentrated in degrees $> -k$.

The curvature map is the quadratic polynomial

$$\Phi (\mu) = d\mu + \mu^2 : A^1 \to A^2$$

The Maurer-Cartan locus

$$\text{MC} (A) = V (\Phi) \subset A^1$$

is the zero locus of $\Phi$.

Given $\mu, \nu \in \text{MC} (A)$, define a differential $d_{\mu, \nu}$ on the graded vector space underlying $A$ by

$$a \in A^i \mapsto d_{\mu, \nu} a = da + \mu a - (-1)^i \nu a \in A^{i+1}.$$  

Let $C^\bullet (\Delta^n)$ be the differential graded algebra of normalized simplicial cochains on $\Delta^n$ with coefficients in $K$.

An element $a \in C^\bullet (\Delta^n) \otimes A^\bullet$ corresponds to a collection of elements

$$\left( a_{i_0 \ldots i_k} \in A^{i-k} | 0 \leq i_0 < \cdots < i_k \leq n \right)$$

where $a_{i_0 \ldots i_k}$ is the evaluation of the cochain $a$ on the face of the simplex $\Delta^n$ with vertices $\{i_0, \ldots, i_k\}$.

The differential on the differential graded algebra $C^\bullet (\Delta^n) \otimes A$ is the sum of the simplicial differential on $C^\bullet (\Delta^n) \otimes A$ and the internal differential of $A$:

$$(\delta a)_{i_0 \ldots i_k} = \sum_{l=0}^k (-1)^l a_{i_0 \ldots i_l \ldots i_k} + (-1)^k d (a_{i_0 \ldots i_k})$$

The product of $C^\bullet (\Delta^n) \otimes A$ combines the Alexander-Whitney product on simplicial cochains with the product on $A$: if $a$ has total degree $j$, then

$$(a \cup b)_{i_0 \ldots i_k} = \sum_{l=0}^k (-1)^{(j-l)(k-l)} a_{i_0 \ldots i_l} b_{i_l \ldots i_k}.$$
The nerve of a differential graded algebra $A$ is the simplicial scheme $N \cdot A$ such that $N_n A$ is the Maurer-Cartan locus of $C^\bullet (\Delta^n) \otimes A$:

$$N_n A = \text{MC} (C^\bullet (\Delta^n) \otimes A)$$

**Theorem 5.1.** Let $A$ be a differential graded algebra such that $A^i$ is finite-dimensional for $i \leq 1$, and vanishes for $i \leq -k$. Then $N \cdot A$ is a (regular) $k$-category.
References
