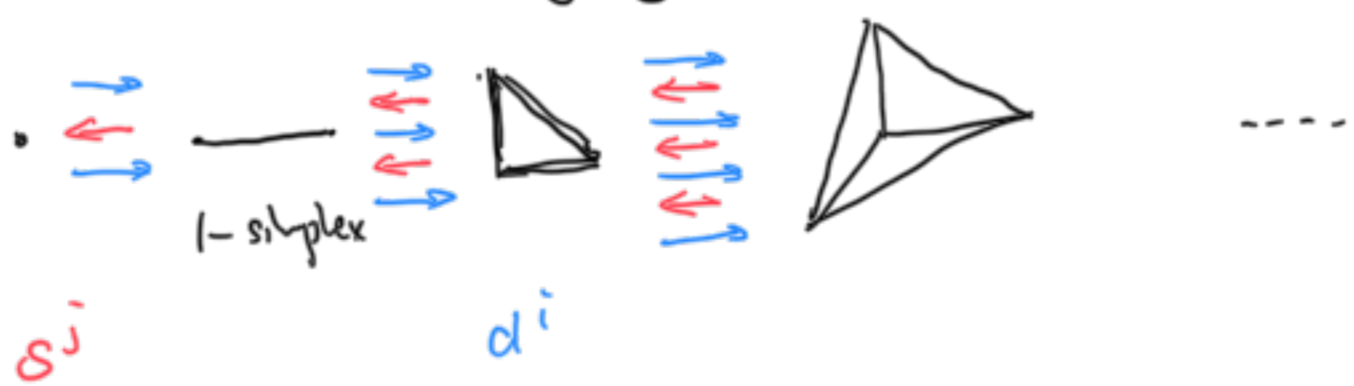
Last time

1. Intuition & definition of $\underline{\Omega_{+|-}}$ ↙ relative
2. Ideas of defining $L_{-|-}$.
   and André – Quillen (co) homology

Goal To give formal definition of $\underline{L_{-|-}}$, with some examples. and explore some properties.

Def The category of Δ



Def Fix some comm. alg k. a simplicial k-alg is a functor $A_* : \Delta^{op} \longrightarrow k\text{-Alg}$,

$$A_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} A_1 \;\overset{d_0}{\underset{d_2}{\overset{d_1}{\leftarrow\!\!\leftarrow\!\!\leftarrow}}}\; A_2 \;\leftarrow\!\!\leftarrow\!\!\leftarrow\!\!\leftarrow_{d_0, d_1, d_2, d_3}\; A_3 \;\cdots$$

$$d_0 - d_1 \qquad d_0 - d_1 + d_2 \qquad d_0 - d_1 + d_2 - d_3$$

Def A semi-free simplicial k-Alg is a simplicial k-alg P_* satisfying

(ii) $s_j : P_{n-1} \to P_n$ $\qquad s_j(X_{n-1}) \subseteq X_n$

<u>Example</u> $\quad k = \mathbb{C}$

$$k[x] \;\substack{\xleftarrow{id} \\ \xrightarrow{id} \\ \xleftarrow[id]{}}\; k[x] \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; k[x] \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}$$

this is <u>semi-free</u>

<u>Def</u> Given any (simplicial) $R \to S$. a simplicial resolution is

$$R \hookrightarrow \underline{P_*} \twoheadrightarrow S$$

s.t. (i) P_* is a semi-free simplicial R-alg

(ii) The composition is $R \to S$

(iii) $R \hookrightarrow P_*$ is injective (cofibration)

$$R \hookrightarrow P_0 \quad - \;—\; —$$

(iv) $P_* \twoheadrightarrow S$ is surjective (fibration)

$$H_0(P_\cdot) = S \qquad H_i(P_\cdot) = 0 \text{ for } i > 0$$

Def

$$\underline{L_{S/R} = \Omega_{P_*/R} \otimes_{P_*} S}$$

is the cotangent complex of $R \to S$

Example $\boxed{\mathbb{C} \longrightarrow \mathbb{C}[x]}$

$$\begin{array}{c} \downarrow\uparrow\downarrow\uparrow\downarrow \\ \mathbb{C}[x] \\ d_0 \downarrow\uparrow\downarrow d_1 \\ \mathbb{C} \hookrightarrow \mathbb{C}[x] \xrightarrow{id} \mathbb{C}[x] \end{array}$$

$$H(\mathbb{C}[x]_*) \simeq \underline{\mathbb{C}[x]}[0]$$

$\boxed{\text{Example}}$ $\boxed{\mathbb{C}[x,y]}/\underline{(y^2-x^3)} \longleftarrow \mathbb{C}$

$$\begin{tikzcd}
& \mathbb{C}[x,y,z_1] \arrow[d, shift right=4] \arrow[d] \arrow[d, shift left=4] & \\
& \mathbb{C}[x,y,z_1] \arrow[u, shift right=2, "\mathrm{id}"'] \arrow[u, shift left=2] \arrow[d, shift right=3, "d_0"'] \arrow[d, shift left=3, "d_1"] & \\
\mathbb{C} \arrow[r, hook] & \mathbb{C}[x,y] \arrow[u, "s_0"] \arrow[r, two heads, "/f"] & \mathbb{C}[x,y]/y^2-x^3
\end{tikzcd}$$

$\underline{\ker(d_0-d_1)} = 0$

$s_0 = x \longmapsto x$
$\quad\;\; y \longmapsto y$

$$H_0 = \frac{\mathbb{C}[x,y]}{\mathrm{im}(d_0-d_1)} = \langle y^2-x^3\rangle$$

$d_0:\ x \longmapsto x$
$\quad\;\; y \longmapsto y$
$\quad\;\; z_1 \longmapsto 0$

$d_1:\ x \longmapsto x$
$\quad\;\; y \longmapsto y$
$\quad\;\; z_1 \longmapsto y^2-x^3$

In general one should use bar construction to get a resolution

$$\rightsquigarrow\ L_{\mathbb{C}[x,y]/(y^2-x^3)/\mathbb{C}} = \Omega_{P_*/\mathbb{C}} \otimes_{P_*} \mathbb{C}[x,y]/f$$

$\uparrow$

$H_i = 0 \quad i \geq 2$

<u>Prop</u> The def of $L_{S/R}$ is independent of the choice of the semi-free resolution

$$
\begin{tikzcd}
 & P_* \arrow[dr, two heads, "\sim"] \arrow[d, shift left, "\sim"] & \\
R \arrow[ur, hook] \arrow[dr, hook] & Q_* \arrow[u, shift left, "\sim"] \arrow[r] & S
\end{tikzcd}
$$

<u>Def</u> For a comm. ring R. if $M = R$-mod.

$\underline{\{x_1, \dots, x_n\}} \subseteq M$. s.t

$n=1$

(i) $(x_1, \dots, x_n) \neq M$

(ii) x_i is a nonzero-divisor in $M/(x_1, \dots, x_{i-1})$

$(x_1, \dots, x_n)$ is a regular sequence

<u>Def</u> A ring map $\varphi : R \to S$ is a <u>complete intersection</u> if φ is surjective and

$$\ker \varphi = (x_1, \dots, x_n)$$

geometrially hypersurface transversely
pass through "each other"

<u>Prop</u> $\varphi : R \to S$ ring homomophism
between <u>Noetherian</u> rings then

TFAE

(i) φ is complete intersection (c.i.)

Tor
Ext

(ii) $D_n(S/R, -) = 0 \quad n \geq 2$

(iii) $D_2(S/R, -) = 0$

$$D_n = H_n(L_{S/R} \otimes M)$$

<u>Properties</u>

(i) Normalization

$$D_0(S/R, M) = \Omega_{S/R} \otimes_S M$$

(ii) base change

$$R \xrightarrow{f} S$$

$$L_{S/R} \otimes (S \otimes_R A) \longrightarrow L(S \otimes_R A / A)$$

this map is a w.e. if either f, α is flat.

<u>Def</u> $\cdots \to C_n \to C_{n-1} \to C_{n-2} \to \cdots$

$\mathrm{fd}\, C_\cdot$ = smallest n. s.t

$$\cdots \to 0 \to \underbrace{F_n \to F_{n-1} \to \cdots F_0}_{\text{free}} \to 0$$

$$F_\cdot \simeq C_\cdot$$

<u>Prop</u> Let R = local ring

$k = R/m$

then TFAE

(i) R is c.i.

(ii) $D_n(k/R; -) = 0$ for $n \geq 3$

Conjecture (Quillen)

if $\quad fd_S \, L_{S/R} < +\infty$

then $\quad \underline{fd_S} \, L_{S/R} \leq 2$

Example

$$\overset{R}{\mathbb{C}} \hookrightarrow \overset{S}{\mathbb{C}[x,y]/\underline{(y^2, xy, x^2)}}$$

$$fd_S \, L_{S/R} = +\infty$$