Introduction to Embedding Calculus

Manifold : M (Space X)

$\mathcal{O}$: poset category of open sets of M (or X)

$F : \mathcal{O}^{op} \to$ spaces - functor

$V_0, V_1 \in \mathcal{O}$ Mayer Vietoris Property

$$\begin{tikzcd} V_0 \cup V_1 & V_0 \arrow[l, hook'] \\ V_1 \arrow[u, hook] & V_0 \cap V_1 \arrow[l, hook'] \arrow[u, hook] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(V_0 \cup V_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(V_0) \arrow[d] \\ F(V_1) \arrow[r] & F(V_0 \cap V_1) \end{tikzcd}$$

$\mathrm{Map}(-, Y)$ satisfies it.

$\mathrm{Imm}(-, N)$

$$\begin{tikzcd} \mathrm{Imm}(V_0 \cup V_1, N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Imm}(V_0, N) \arrow[d] \\ \mathrm{Imm}(V_1, N) \arrow[r] & \mathrm{Imm}(V_0 \cap V_1, N) \end{tikzcd}$$

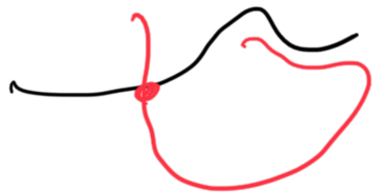
Homotopy pull back square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \quad \text{if} \quad A \simeq \mathrm{holim}\left(\begin{tikzcd} & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \right)$$

$\mathrm{Imm}(-, N)$ preserves homotopy pullbacks

(Smale, Haefliger-Poenaru)

$\mathrm{Emb}(-, N)$ does not



Let

$$V = V_0 \cup V_1$$

$$A_0 = V \setminus V_0 \qquad A_0, A_1 \subseteq V \text{ closed disjoint subsets of } V$$

$$A_1 = V \setminus V_1$$

$$F(V) \simeq \operatorname{holim} \left(\begin{tikzcd} & F(V \setminus A_1) \arrow[d] \\ F(V \setminus A_0) \arrow[r] & F(V \setminus (A_0 \cup A_1)) \end{tikzcd} \right)$$

<u>Weaker property</u>

Let $V \in \mathcal{O}$ $\quad A_0, \dots A_k \subseteq V$ pairwise disjoint closed

Let $\mathrm{Cube}_F(V, A_0, \dots A_k)$ be the punctured cube diagram

$(k=2)$

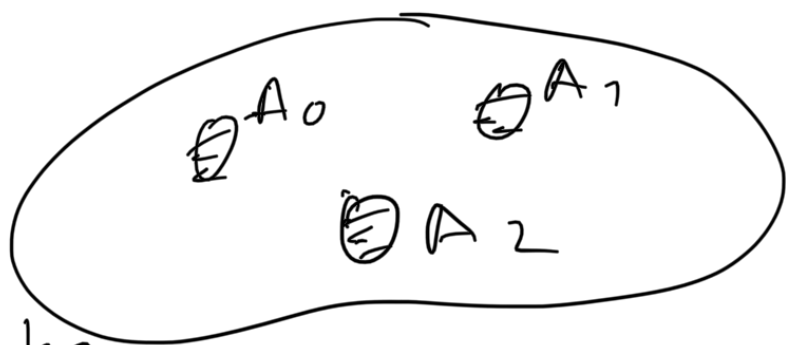
$$\begin{tikzcd}
 & F(V \setminus A_1 \cup A_2) \arrow[rr] & & F(V \setminus A_0 \cup A_1 \cup A_2) \\
F(V \setminus A_1) \arrow[ur] \arrow[rr] & & F(V \setminus (A_0 \cup A_1)) \arrow[ur] & \\
 & F(V \setminus A_2) \arrow[uu, dashed] \arrow[rr] & & F(V \setminus A_0 \cup A_1) \arrow[uu] \\
 & & F(V \setminus A_0) \arrow[ur] &
\end{tikzcd}$$

__Definition__: F is __polynomial__ of degree $\leq k$

if $\forall V \in \mathcal{O}$, $A_0, \cdots A_k \subseteq V$

$$F(V) \simeq \operatorname{holim}(\mathrm{Cube}_F(V, A_0 \cdots A_k)).$$

V



Punch out A_0, A_1, A_2

__Examples__

$\mathrm{Map}(-, X)$: Polynomial deg ≤ 1

$\mathrm{Imm}(-, N)$: deg ≤ 1

$\mathrm{Map}(\mathrm{UConf}(-, k), X)$: deg $\leq k$

$\mathrm{Emb}(-, N)$: Not poly. for any k

$$\mathrm{Emb}(-, N) \longrightarrow \mathrm{Imm}(-, N)$$

deg 1 appox

Theorem: (Goodwillie - Weiss)

Any "good" functor F has a degree k approximation $T_k F : \underline{\mathcal{O}}^{op} \to \underline{\text{Spaces}}$ such that $T_k F$ is polynomial of degree $\leq k$

Properties:

1) $\eta_k : F \longrightarrow T_k F$

2) η_k is a w.e if F is poly $\leq k$

3) Let $\eta : F \longrightarrow G$ $\quad F, G$ poly $\leq k$

$\mathcal{O}_k \subset \mathcal{O}$ $\quad \mathcal{O}_k := \{ W \in \mathcal{O} \mid W \cong$ disj union of upto k balls $\}$

$$F|_{\mathcal{O}_k} = G|_{\mathcal{O}_k}$$

then η is a w.e.

4) For any good functor F

$$F|_{\mathcal{O}_k} \cong T_k F|_{\mathcal{O}_k}$$

Define $T_k F(V)$ by extending via holim

$$T_k F(V) = \underset{\substack{W \subseteq V \\ W \in O_k}}{\operatorname{holim}} \, F(W)$$

5) Poly $\leq k$ is poly of deg $\leq k+1$

Look at punctured cube diagram, if F was poly of degree ≤ 1, all pull back squares agree. ...

Calculus	Embedding Calculus
Continuous functions	Good Functions
Polynomials	polynomial functions
Taylor Approx.	$F \to T_k F$
If f is poly $\leq k$ is determined by $f(x_i)$ $i \in \{0, \dots k\}$	If F is poly $\leq k$ F is determined by $F\|_{O_k}$

"Taylor Approximation"

For a function:

$f(x)$: Polynomials $T_0 f(x)$, $T_1 f(x)$, $T_2 f(x)$.

such that $f(X) \simeq \lim_{i \to \infty} T_i f(X)$

<u>For Functors:</u>

$$F \to T_k F \qquad \forall k$$

Note $T_{k+1} F \to T_k F$ because $\underline{\mathcal{O}_k} \subset \mathcal{O}_{k+1}$

We get a tower:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & T_3 F \arrow[d] \\
 & T_2 F \arrow[d] \\
F \arrow[r, "\eta_1"] \arrow[ur, "\eta_2"] \arrow[uur, "\eta_3"] & T_1 F
\end{tikzcd}$$

like Postnikov Towers

$$T_\infty F := \operatorname{holim}_i \, (T_i F)$$

Under <u>nice</u> conditions $F \xrightarrow{w.e} T_\infty F$

<u>Example</u>: $F = \mathrm{Emb}(-, N)$

If $\dim N - \dim M \geq 3$. $\boxed{F \simeq T_\infty F}$

Furthermore

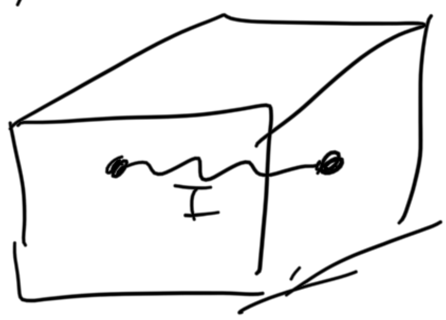
$$\eta_k : \underline{\mathrm{Emb}(M, N)} \longrightarrow \underline{T_k \mathrm{Emb}(M, N)}$$

$k(\dim N - \dim M - 2) + 1 - \dim M$ connected

$\geq k \cdot 1 + 1 - \dim M$

As $k \uparrow$ $\quad \mathcal{T}_k$ is more connected

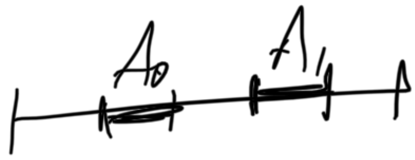
Ex $\boxed{\mathrm{Emb}_\partial(I, M)}$: Embeddings $I \to M$ fixing $0, 1 \in \partial M$



$\dim M \geq 4$

Goal:
Finite model for $T_k \mathrm{Emb}(I, M)$:
Start with

$T_1 \mathrm{Emb}_\partial(I, M)$



$$\simeq \operatorname{holim} \left(\begin{array}{ccc} & & \forall\ T_1 \mathrm{Emb}(\vdash\!\!-\ -\!\!\dashv, M) \\ & & \downarrow \\ T_1(\mathrm{Emb}(\vdash^{A_0} \!\!-\!\!\dashv, M)) & \xrightarrow{*} & T_1(\mathrm{Emb}(\vdash_{A_0} -\ -\!\!\dashv_{A_1}, M)) \end{array} \right)$$

$$\mathrm{Emb}(\bullet\!\!\longleftarrow\ \longrightarrow\!\!\bullet, M) \simeq \mathrm{Emb}(\bullet\ \ \bullet, M) \cong *$$

Fixing end points

Note

$$T_1 \mathrm{Emb}(\vdash\!\!-\!\!-\!\!\dashv, M) \simeq T_1 \mathrm{Emb}(\phi \quad \phi, M) \simeq *$$

Also,

$$T_1 \mathrm{Emb}(\phi\!\leftarrow \;-\; \rightarrow\!\phi, M)$$

$$\cong$$

$$T_1 \; \mathrm{Emb}_\partial(\phi \; \underset{\substack{\cap \\ I \,=\, B^1 \,\in\, O_1}}{-\!\!-} \; \phi, M)$$

$$\cong$$

$$\mathrm{Emb}(-, M) \dashrightarrow \left(\begin{array}{c} T_1\big(\mathrm{Emb}(-, M)\big) \\ = \mathrm{Emb}(-, M)\big) \end{array} \right)$$

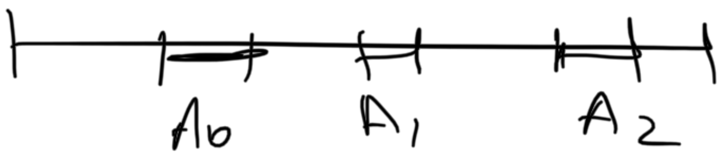
$$\underline{T_1 \mathrm{Emb}_\partial(I, M)} = \operatorname{holim} \left(\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & \mathrm{Emb}(-, M) \end{tikzcd} \right)$$

$$= \Omega \, \mathrm{Emb}(I, M)$$

$$= \Omega \, STM \simeq \underline{\mathrm{Imm}_\partial(I, M)}$$

STM: sphere tangent bundle of M

$T_2\, \mathrm{Emb}_\partial(I, M)$



$T_2(I \setminus A_0)$: $T_2(\quad)$

|2

$*$

$T_2(I \setminus (A_0 \cup A_1)) = T_2(\quad)$

$= T_2(\quad)$ $I_1 \in \mathcal{O}_1 \subset \mathcal{O}_2$

$= \mathrm{Emb}(I, M)$

(not fixing boundary)

$T_2(I \setminus (A_0 \cup A_1 \cup A_2))$

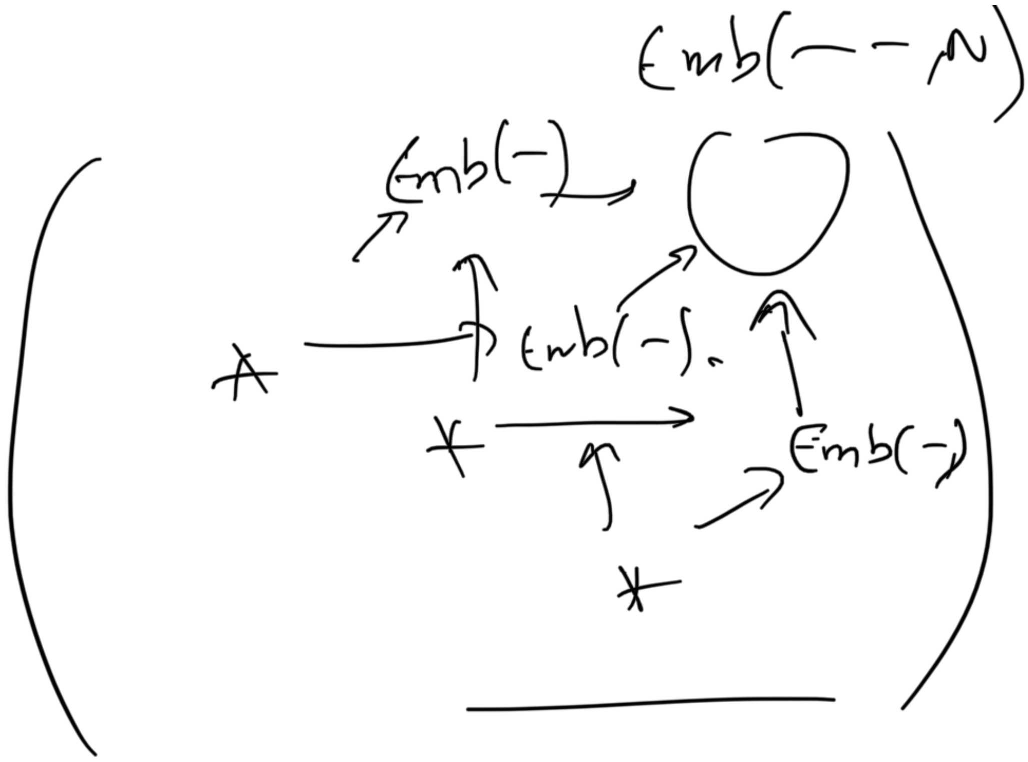
$= T_2(\quad)$

$= T_2(\quad)$

$I_1 \cup I_2 \in \mathcal{O}_2$

$= \mathrm{Emb}(I_1 \cup I_2, M)$

$$T_2\,\mathrm{Emb}_\partial(I,M) = \underline{\mathrm{holim}}$$



$T_2\,\mathrm{Emb}_\partial(I,M)$ is a finite homotopy limit of $\mathrm{Emb}(\underline{\sqcup} B^1, M)$

$T_k\,\mathrm{Emb}_\partial(I,M)$ is also a finite htpy limit

$= \mathrm{holim}$ (punctured $k+1$ cube)

$$\underline{\underline{\subseteq}} \quad \mathrm{Maps}\left(\Delta^k, \mathrm{Emb}\left(\bigsqcup_{i=1}^{k} B^1, M\right)\right)$$