(prep) Homotopy Limits

Wednesday, April 14, 2021 11:27 PM

Limits

For $D : \mathcal{D} \to \mathrm{Set}$, $\lim_{\mathcal{D}} D$ is the set

$$\{ (x_d)_{d \in \mathrm{Ob}(\mathcal{D})} \mid x_d \in Dd,\ Df(x_d) = x_{d'}\ \forall f : d \to d' \text{ in } \mathcal{D} \} \subseteq \prod_{d \in \mathrm{Ob}(\mathcal{D})} Dd$$

Ex (pullbacks)

$$\mathcal{D} = \begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \qquad D = \begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

$$\lim_{\mathcal{D}} D = \{ (a,b,c) \in A \times B \times C \mid fa = gb = c \} \cong \{ (a,b) \in A \times B \mid fa = gb \}$$

Ex $\mathcal{D} = \begin{tikzcd} \cdot \arrow[r] \arrow[rr, bend right] & \cdot \arrow[r] & \cdot \end{tikzcd}$ $\quad D = \begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & B \arrow[r, "g"] & C \end{tikzcd}$ $\quad \lim_{\mathcal{D}} D = \{ (a,b,c) \in A \times B \times C \mid fa = b,\ gb = c,\ gfa = c \}$

$$\cong \{ (a, fa, gfa) \} \cong A$$

Homotopy limits

Like limits, but replace $=$ with $\sim$ (coherently)

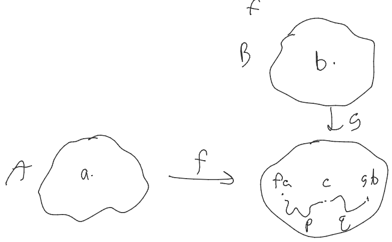
Recall $N\mathcal{D} : \Delta^{op} \to \mathrm{Set}$, $N\mathcal{D}_n = \{ d_0 \xrightarrow{f_1} d_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} d_n \}$ $\quad N\mathcal{D}_0 \cong \mathrm{Ob}\,\mathcal{D}$

For $D : \mathcal{D} \to \mathrm{Space}$ (= Top, sSet, any simplicial model cat)

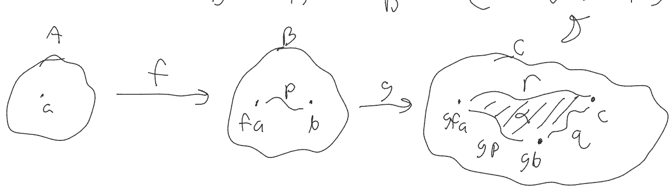
$\mathrm{holim}_{\mathcal{D}} D$ is the space

$$\Big\{ \Big((x^0_d \in Dd)_{d \in N\mathcal{D}_0},\ (x^1_{f : d \to d'} \in \mathrm{Path}_{Dd'}(Df(x^0_d), x^0_{d'}))_{f \in N\mathcal{D}_1},\ (x^2_{d \xrightarrow{f} d' \xrightarrow{g} d''} \in \mathrm{Triangle}(Dg(x^1_f), x^1_g, x^1_{gf}))_{(f,g) \in N\mathcal{D}_2}, \ldots \Big) \Big\}$$

Ex $\mathcal{D} = \begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ $\quad D = \begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$ $\quad \mathrm{holim}_{\mathcal{D}} D = \{ (a,b,c,p,q) \}$

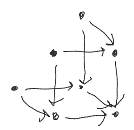
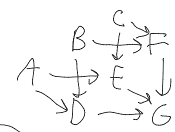
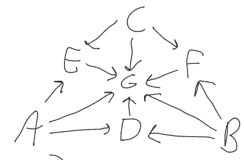


Ex $\mathcal{D} = \cdot \to \cdot \to \cdot$ $\quad D = A \xrightarrow{f} B \xrightarrow{g} C$ $\quad \mathrm{holim}_{\mathcal{D}} D = \{ (a,b,c,p,q,r,\alpha) \}$

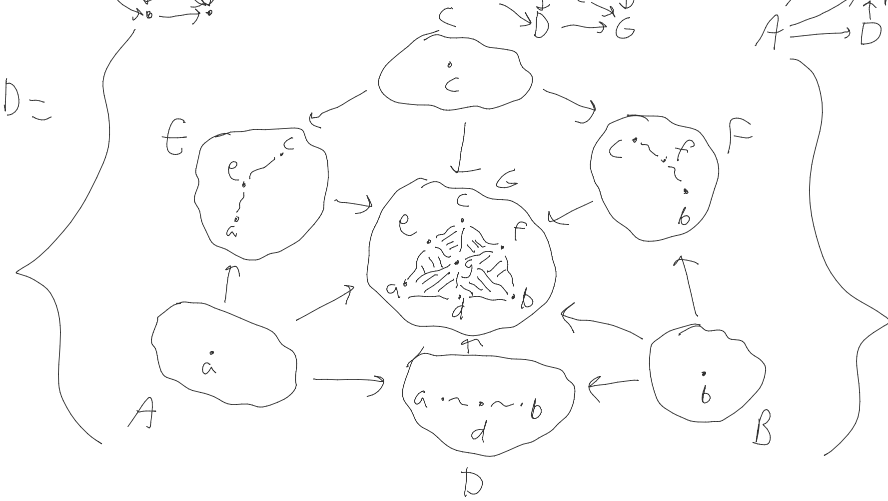


Ex $\mathcal{D} =$ $\quad D = $

$$\begin{tikzcd} & B \arrow[r] \arrow[d] & F \\ A \arrow[r] & E & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ E \arrow[r] & G & F \arrow[l] \end{tikzcd}$$

Ex $\mathcal{D}$ =  D =  

$\operatorname{holim}_{\mathcal{D}} D =$



A, B, C: $\cdot$ $\cong |N\mathcal{D}/A|$ (or B, C)

D, E, F: $\cdot\!-\!\!\cdot\!\!-\!\cdot$ $\cong |N\mathcal{D}/E|$ (or D, F)

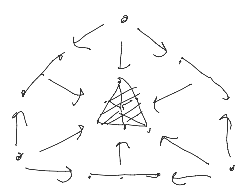
G  $\cong |N\mathcal{D}/G|$

Formalism

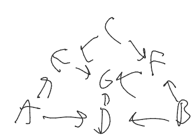
For $D : \mathcal{D} \to$ Space, what does $\operatorname{holim}_{\mathcal{D}} D$ need from each space Dd?

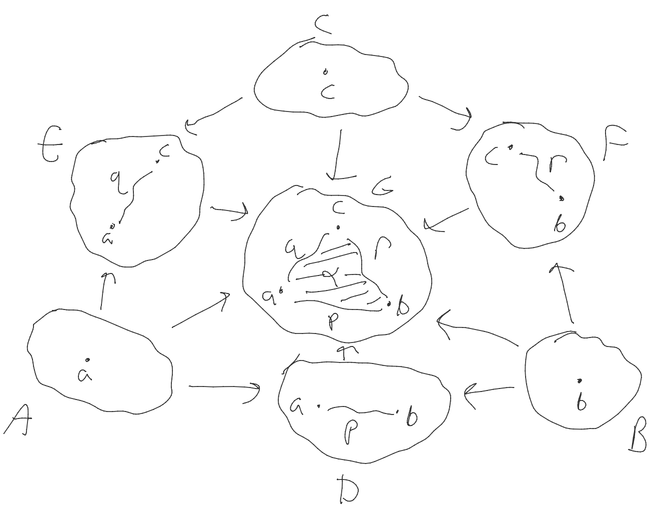
Let $\text{shape}_{\mathcal{D}} : \mathcal{D} \to \text{Space}$

$d \mapsto |N\mathcal{D}/d|$



Then $\boxed{\operatorname{holim}_{\mathcal{D}} D \cong \operatorname{Nat}(\text{Shape}_{\mathcal{D}}, D)}$ (one of many formal defns)

Reduction: holim  $\cong \{(a, b, c, p, q, r, \alpha)\}$



<u>Ex</u> If all the maps are inclusions, then
$\operatorname{holim} = \{ \text{triangles } \alpha \text{ in } G \mid \text{edges in } D, E, F, \text{ vertices in } A, B, C \}$

property, not structure

<u>Properties</u>

- (Homotopy invariance) If $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "D"] \arrow[r, bend right, "D'"'] \arrow[r, phantom, "\Downarrow s"] & \text{space} \end{tikzcd}$ is a natural weak equivalence, then $\operatorname{holim}_{\mathcal{D}} D \xrightarrow{\sim} \operatorname{holim}_{\mathcal{D}} D'$
(unlike lim in general)

- (universal property)

$$\begin{tikzcd} & & \operatorname{holim}_{\mathcal{D}} D \arrow[r] \arrow[dr] \arrow[ddr] & Dd \arrow[d] \arrow[dd, bend left] \\ & & & Dd' \arrow[d] \\ * \arrow[r] \arrow[uurr, dashed, "\exists!"] & E \arrow[uur, dashed] \arrow[uurr] \arrow[urr] \arrow[rr] \arrow[urr, phantom, "\wr s"] & & Dd'' \end{tikzcd}$$

is the terminal homotopy cone of D

- (strict limit comparison)
If D is an "injective fibrant" diagram, $\operatorname{holim} D \simeq \lim D$.
no concrete description known in general

The cone $\begin{tikzcd} \lim D \arrow[r] \arrow[dr] & Dd \arrow[d] \arrow[l, phantom, "\wr"] \\ & Dd' \end{tikzcd}$ induces $\lim D \to \operatorname{holim} D$ as does

$\lim_{\mathcal{D}} D \cong \operatorname{Nat}(*, D)$ $\qquad *: \mathcal{D} \to \text{Space}$

$\qquad\qquad\qquad\qquad\qquad\qquad d \mapsto *$

$\operatorname{holim}_{\mathcal{D}} D \cong \operatorname{Nat}(\operatorname{Shape}_{\mathcal{D}}, D)$