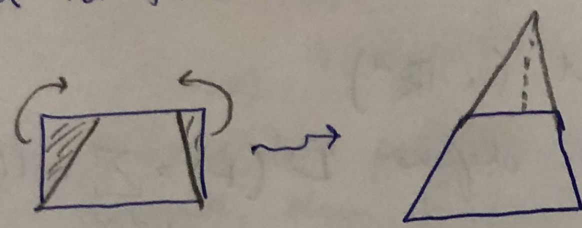
Idea: when can we take two polytopes, and cut one up, and rearrange the pieces so that its equal to the other?

Example:



Answer: in 2-dim Euclidean, this occurs exactly when the two polytopes have the same area

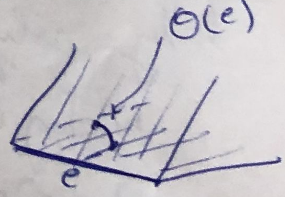
Def: Given two polytopes P and Q, they are scissors congruent if $P = \bigcup_{i=1} P_i$ and $Q = \bigcup_{j=1}^{m} Q_j$, and $P_i \cong Q_i$, and $\text{mea}(P_i \cap Q_i) = 0$.

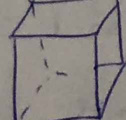
Question: What about higher dimensions?
In 3-dim, volume is not the only invariant:

Def: Dehn invariant (in $\mathbb{R}^3$)

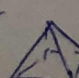
$$D(P) = \sum \text{len}(e) \otimes \Theta(e) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$$

where $\Theta(e)$ is the angle at e, defined as the proportion of S^1 inside the polytope



Example: D  $= \sum_{12} 1 \otimes_{\mathbb{Z}} \frac{\pi}{2} \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}\pi$

$= 1 \otimes_{\mathbb{Z}} 6\pi = 0$

D  $\neq 0$

Upshot: 2-dim : area seperates s.c. classes
3-dim : vol and D seperate s.c. classes

Q: Higher dim?

Def: Dehn invariant (in $\mathbb{R}^d$)
Let X^n be n-dim, define $D^i(P) = \sum_{\substack{i\text{-dim} \\ \text{face of } P,\, f}} \text{vol}(f) \otimes \Theta(f)$

where $\Theta(f)$ is proportion of sphere at face f

Generalized Hilbert's 3'd Problem:
In Euclidean, spherical, and hyperbolic geometries, do the volume and generalized Dehn invariant seperates scissors congruence classes of polytopes?

Remark: • we can think of D^i as a map from polytopes to $\mathbb{R} \otimes \mathbb{R}/_{\mathbb{Z}\pi}$
• we can give polytopes, i.e. domain of D^i a group structure:

Def: Let X be a space, then P(X) is the group generated by polytopes, modulo relations ← isometry group of X
1) $[gQ] = [Q]$ for all $g \in I(X)$
2) $[Q \cup P] = [Q] + [P]$ if $P \cap Q$ is contained in a finite union of m-1 simplicies

Remark: a polytope P is defined to be a finite union of m-simplicies

Thus: $D^i : P(X) \longrightarrow \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/_{\mathbb{Z}\pi}$

Now, we can realize $P(X) \xrightarrow{V} \mathbb{R} \otimes \mathbb{R}/2\pi$
$\cup$
$\mathrm{Ker}(D)$

And thus, we get the modern phrasing of the problem, asking if $V|_{\ker(D)}$ is injective.

> Aside: suppose two polytopes P and Q have same Dehn invariant, then $P - Q \in \ker(D)$.
> But if $V|_{\ker(D)}$ is injective, then $V(P-Q) = 0$
> $\Leftrightarrow P = Q$

Conjecture: (Goncharov)

Solve this by constructing a map

$$\underbrace{\ker D \otimes \mathbb{Q} \hookrightarrow \left(gr_n^{\gamma} K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \otimes (\mathbb{Q}^{\sigma})^{\otimes n}\right)^{+} \xrightarrow{?} \mathbb{R}/2\pi\mathbb{Q}}_{\mathrm{Vol}}$$

borel regulator

such that

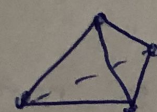
Inna and Johnathan proved a version of this for S^n and $\mathcal{H}^n$

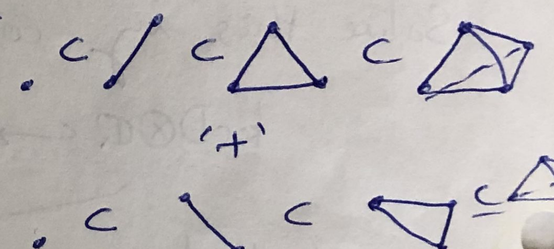
Namely, in spherical, they prove:

Thrm: $\ker D \hookrightarrow H_d(O(d+1;\mathbb{R}), \mathbb{Z}[\tfrac{1}{2}])$ which after composition w/ "cheegr-Chern-Simons" class is equal to the volume map

Summary: translate:
- polytopes $\rightsquigarrow$ RT-building (simplicial set)
- $P(X, I(X)) \rightsquigarrow$ homotopy coinvariants
- form "Deh complex"
- "do the algebra on the spaces first"
  (e.g. replace $\otimes \rightsquigarrow$ reduced join)

Section:
polytopes $\rightsquigarrow$ RT-building (simplicial set)

Idea: polytope  $\rightarrow$ flag



since polytope can be thought of as a union of simplicies, replace by chains of simplicies and replace simplex by subspace

Def: Let X be dim n. Define $T_\bullet^m(X)$ be the simplicial set whose i-simplicies are sequences $U_0 \subseteq \dots \subseteq U_i$ of non-empty subspaces of X of dim at most m.
and jth face deletes U_j
jth deg repeats U_j

An RT-building is $F_\bullet^X = T_\bullet^n(X) / T_\bullet^{n-1}(X)$

so i-simplices are non-empty seq
$U_0 \subset \dots \subseteq U_i$ s.t. $U_i = X$

Next: realize $P(X)$ as homotopy coinvariants:

Df: For a G-module M, the coinvariants of $G \curvearrowright M$ is

$$M / (m - g\cdot m \mid g \in G, m \in M)$$

Then, we see in df of $P(x)$, we are quotienting by $[P] - g\cdot[P]$, so we have $I(x) \curvearrowright P(X, 1)$ and get

Lemma 1: if $G < I(x)$, then

$$P(X, G) \cong H_0(G, P(X,1))$$

* Lemma 2: $P(x,1) \cong H_n(F_\bullet^x)^{\sigma} \cong H_{n+1}(\underbrace{S^0 \wedge F_\bullet^x}_{\text{adding in a twist}})$

~~Lemma 3~~: $\cancel{H_n(F_\bullet^x) \cong H_{n+1}(\underbrace{S^0 \wedge F_\bullet^x}_{\text{adding in a twist}})}$

as groups, while as $I(X)$-modules, differ by action of S^0 which is a twist $*^{\sigma}$ by the determinant

Thrm: $P(X, G) \cong H_0(G, P(X,1)) \cong H_0(G, H_n(F_\bullet^X)^{\sigma})$

$$\cong H_0(G, H_{n+1}(S^0 \wedge F_\bullet^X)) \cong H_{n+1}((S^0 \wedge F_\bullet^X)_{hG})$$

Pf: most follows from something called the "homotopy orbit spectral sequence" and map $P(X,1) \longrightarrow H_{n+1}(S^0 \wedge F_\bullet^X)$

take $\underset{\text{simplex}}{[x_0, \dots, x_n]} \longmapsto \sum_{\sigma \in \Sigma_{n+1}} \operatorname{sgn}(\sigma)[\operatorname{span}(x_{\sigma(0)}) \subseteq \operatorname{span}(x_{\sigma(0)}, x_{\sigma(1)}) \subseteq \dots \subseteq \operatorname{span}(x_{\sigma(0)}, \dots, x_{\sigma(n)})]$