Notes on Fibrations in Type Theory

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1 Motivation

1. We said that we want to think of type families as fibrations
   (a) That is, we should be able to lift from paths in the base space to paths in the above space
2. However, it's not clear what this should mean categorically
3. Today: categorical definition of fibration & why we care

2 Type Families as Cat-Valued Functors

1. We can think of $z : \mathbb{N} \vdash \text{Vec}(z) : \mathit{Type}$ as a functor two ways
   (a) $- \vdash - : \mathit{Type}$ takes a context (in this case {z : N}) to a Category

   (b) $z : \mathbb{N} \vdash \text{Vec}(z) : \mathit{Type}$ takes an object z of $[\![\mathbb{N}]\!]$ to a category, and a path in $[\![\mathbb{N}]\!]$ to an equivalence of categories
2. What does the action on morphisms look like in the first case?
   (a) $\text{Vec}(z)[z \mapsto x + y] = \text{Vec}(x + y)$, so $[z \mapsto x + y]$ should correspond to a functor from $[\![\text{Vec}(z)]\!]$ to $[\![\text{Vec}(x + y)]\!]$
   (b) As a morphism of context, $[z \mapsto x + y]$ corresponds to

$$\{x : \mathbb{N}, y : \mathbb{N}\} \xrightarrow{[z \mapsto x+y]} \{z : \mathbb{N}\}$$

   (c) Thus, $- \vdash - : \mathit{Type}$ is a *contravarient* functor $\text{Ctxt}^{\text{op}} \to \text{Cat}$

3. Both are contravarient
   (a) Since $[\![\mathbb{N}]\!]$ is a ∞-groupoid, $[\![\mathbb{N}]\!]^{\text{op}} = [\![\mathbb{N}]\!]$.
   (b) So, both are of the form $\mathcal{C}^{\text{op}} \to \text{Cat}$

3 The Grothendieck Completion

1. Let $F : \mathcal{C}^{\text{op}} \to \text{Cat}$
   (a) Write u^* for the functor $F(u) : Fb \to Fa$ when $u : a \to b$
2. Build the following category:

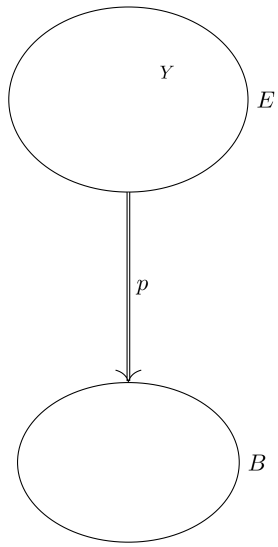
$$\int F = \left\langle \begin{array}{ll} \text{objects :} & (I, X) \text{ where } I \in |\mathcal{C}| \text{ and } X \in |F(I)| \\ \text{morphisms :} & (I, X) \to (J, Y) \text{ are } (u, f) \text{ where u : I} \to J \text{ and f : X} \to u^*(Y) \end{array} \right\rangle$$

3. $\int F$ has all of the structure of the image of F
4. The first projection $\int F \to \mathcal{C}$ projects down to the base category, collapsing the non-$\mathcal{C}$ structure
   (a) It is a (split) fibration

4 The Categorical Definition of Fibration

A functor $p : E \to B$ is a fibration if for every $Y \in \text{Ob}(E)$ and $u : A \to_B pY$ there is an $X \in \text{Ob}(E)$ such that $pX = A$ and a $f : X \to Y$ such that $pf = u$, and for every $g : Z \to_E Y$ such that there is a morphism $w : PZ \to A$ such that $Pg = w; u$, there is a unique $h : Z \to X$ such that $g = h; f$.

1. In diagrams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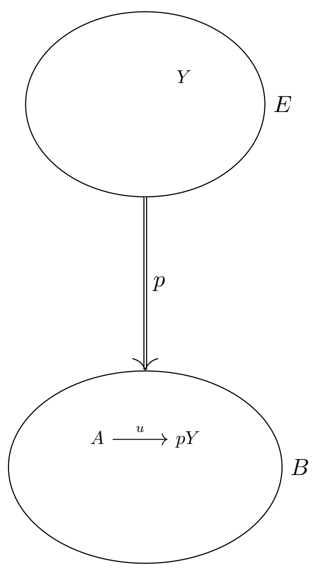
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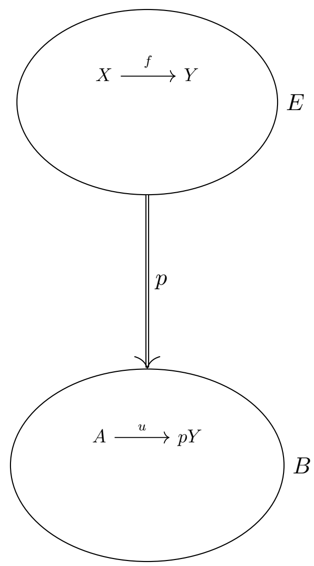
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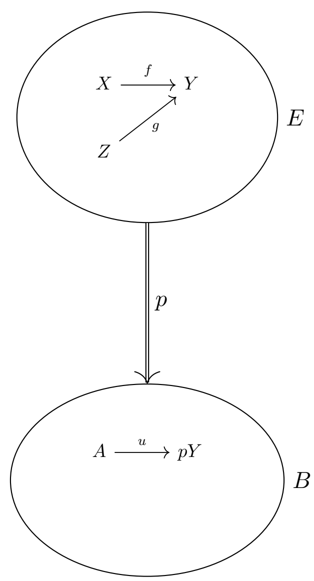
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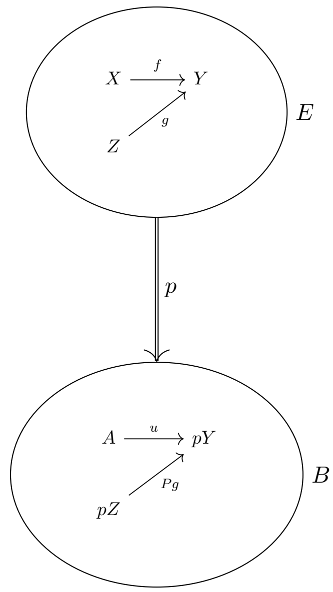
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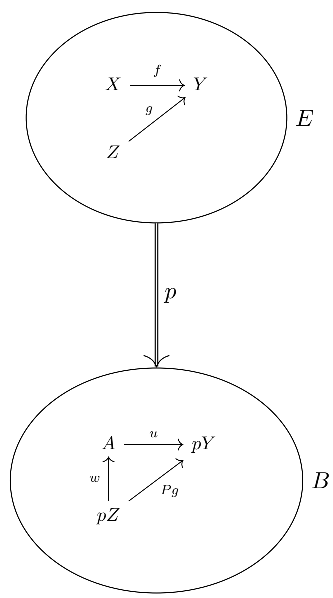
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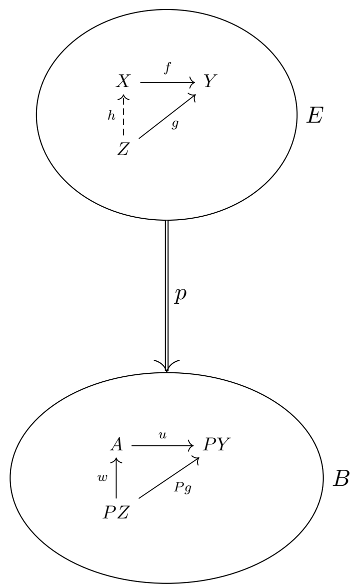
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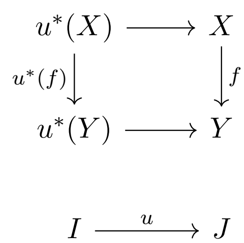
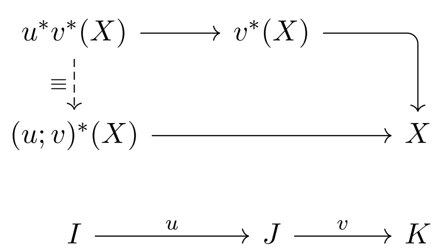
2. Vocabulary
    (a) f is the *cartesian lifting* of u.
    (b) (Any f such that $pf = u$ with the univeral property above is called *cartesian over* u, even if p is not a fibration.)
3. N.B.: Cartesian liftings are not necessarily unique!

5 Split Fibrations

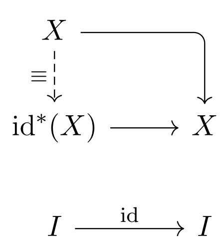
1. Let $p : E \to B$ be a fibration
2. If p has chosen liftings $u^*(X) \to X$ for every $u : I \to J$ and X (above J), then p is *cloven*
3. u^* extends into a functor for every u



4. However, composition and identity don't work out as one might hope
    (a) Composition



    (b) Identity



5. If these equivalences are identities, then the fibration p is *split*

6 The Grothendieck Theorem

1. Grothendieck Completion yields a split fibration
   (a) Let $u : I \to J$ be a morphism in B and (J, Y) be above J in $\int F$
   (b) We have
   $$(I, u^*Y) \xrightarrow{(u,id)} (J, Y)$$
   as a chosen lifting of u
   (c) Because F is a functor, clearly this is split
2. The Theorem
   (a) The category of indexed categories is called ICat
   (b) The category of split fibrations is called $\text{Fib}_{\text{split}}$
   (c) Theorem Statement
   $$\begin{array}{ccc} \text{ICat} & \xrightarrow{\quad\equiv\quad} & \text{Fib}_{\text{split}} \\ & \searrow \quad \swarrow & \\ & \text{Cat} & \end{array}$$
      i. $\int$ extends into $\equiv$
      ii. The other direction comes from finding the fibers of the fibration
         A. I.e., the categories that are above a single object

7 Consequences for Type Theory

1. We can think of type families as fibrations as well as indexed categories
   (a) Properties versus structure
      i. See "Categorical Logic and Type Theory"
2. If we just try to do indexed categories, just end up using Grothendieck everywhere anyways

(a) It's very similar to the space of -types
(b) Might as well just use fibrations from the beginning