Let $\bar{M}$ be a 1-connected $(\mathcal{I}, \mathcal{X})$-manifold. Let us suppose that a group G acts on $\bar{M}$ by $(\mathcal{I}, \mathcal{X})$-maps. Moreover, we assume that the action is topologically free, i.e., every point $\bar{P} \in \bar{M}$ has an open neighborhood U such that $gU \cap U = \emptyset$ for all non-trivial $g \in G$. By (3.4.20), the quotient $G \backslash \bar{M}$ is a $(\mathcal{I}, \mathcal{X})$-manifold.

Let $\bar{\varphi} : \bar{U} \to \mathcal{X}$ be a chart in M. By (3.4.18), the chart extends to a $(\mathcal{I}, \mathcal{X})$-map

$$\tilde{\varphi} : \bar{M} \to \mathcal{X}.$$

For any $g \in G$, the map $\tilde{\varphi} \circ g : \bar{M} \to \mathcal{X}$ is also a $(\mathcal{I}, \mathcal{X})$-map. Since $\bar{M}$ is connected, there is a unique $\xi_g \in \mathcal{I}$ such that

$$\tilde{\varphi} \circ g = \xi_g \circ \tilde{\varphi}$$

(see 3.4.19). Note that, by uniqueness,

$$\xi_g \circ \xi_{h^{-1}} \circ \tilde{\varphi} = \tilde{\varphi} \circ g \circ h^{-1} = \tilde{\varphi} \circ \left(gh^{-1}\right) = \xi_{gh^{-1}} \circ \tilde{\varphi}$$

implies

$$\xi_{gh^{-1}} = \xi_g \circ \xi_{h^{-1}}.$$

Clearly, $\xi_1 = \mathrm{id}_{\mathcal{X}}$. Thus, we have defined a group homomorphism

$$\eta_{\tilde{\varphi}} : G \to \mathcal{I}.$$

Definition 3.4.21. The homomorphism η is called the holonomy of M determined by $\tilde{\varphi} : \bar{M} \to \mathcal{X}$

Note that the construction does not require the map $\tilde{\varphi}$ to be a lift of a chart. For any $(\mathcal{I}, \mathcal{X})$-map $\nu : \bar{M} \to \mathcal{X}$, we obtain a holonomy η_ν.

Exercise 3.4.22. Show that for two $(\mathcal{I}, \mathcal{X})$-maps $\nu_0, \nu_1 : \bar{M} \to \mathcal{X}$, there is a unique $g \in \mathcal{I}$ such that

$$\eta_{\nu_0}(h) = g\eta_{\nu_1}(h)\, g^{-1}.$$

Thus, two holonomies differ by an inner automorphism of $\mathcal{I}$.

Example 3.4.23. If M is path connected, then the universal cover $\tilde{M}$ is a 1-connected manifold. Pull back an atlas for M to put a $(\mathcal{I}, \mathcal{X})$-structure on $\tilde{M}$. Then any chart induces a holonomy

$$\eta : \mathrm{Cov}_M\left(\tilde{M}\right) \to \mathcal{I}.$$

Thus, we can construct geometric representations of the fundamental group of M.

Definition 3.4.24. Let M be a path connected $(\mathcal{I}, \mathcal{X})$-manifold, and let $\delta : \tilde{M} \to \mathcal{X}$ be a $(\mathcal{I}, \mathcal{X})$-map. Then the induced map

$$\eta_\delta : \mathrm{Cov}_M\left(\tilde{M}\right) \to \mathcal{I}$$

characterized by the equation

$$\eta_\delta(\tau) \circ \delta = \delta \circ \tau \qquad \texttt{for all } \tau \in \mathrm{Cov}_M\left(\tilde{M}\right)$$

is called the holonomy associated to δ. That this equation determines a map follows from (3.4.22). It is easy to check that it is a homomorphism.

If we are given a base point $\underline{\mathrm{P}}$ in M, we have an isomorphism

$$\pi_1(M, \underline{\mathrm{P}}) = \mathrm{Cov}_M\left(\tilde{M}\right).$$

Thus, we obtain a homomorphism

$$\eta_\delta : \pi_1(M, \underline{\mathrm{P}}) \to \mathcal{I},$$

which we also call the holonomy.

3.4.3 The Torus

Our goal is to classify Euclidean structures on the torus up to equivalence. It turns out that it is easier to classify them up to similarity. So let us build the set up. Let

- $\mathrm{Homeo}(\mathbb{T})$ be the group of self-homeomorphisms on the torus $\mathbb{T}$, and let
- $\mathrm{Homeo}_1(\mathbb{T})$ be the normal subgroup of those homeomorphisms that are homotopic to the identity. The factor group
- $M(\mathbb{T}) := \mathrm{Homeo}(\mathbb{T})\,/\mathrm{Homeo}_1(\mathbb{T})$ is called the <u>mapping class group</u> of $\mathbb{T}$.

  Let
- $\mathrm{Isom}(\mathbb{E}^2)$ be the isometry group of the plane. This is a normal subgroup in the group
- $\mathrm{Sim}(\mathbb{E}^2)$ of similarities. Note that $\mathrm{Sim}(\mathbb{E}^2)$ acts from the left on
- $\mathcal{E}(\mathbb{T})$, the set of Euclidean structures on $\mathbb{T}$. The action is given by modifying all the charts, appending the similarity $\sigma \in \mathrm{Sim}(\mathbb{E}^2)$. Put
- $\mathcal{S}(\mathbb{T}) := \mathrm{Sim}(\mathbb{E}^2)\,\backslash\mathcal{E}(\mathbb{T})$.

  Note that $\mathrm{Homeo}(\mathbb{T})$ acts on $\mathcal{E}(\mathbb{T})$ from the right as follows: For a homeomorphism $\zeta : \mathbb{T} \to \mathbb{T}$, a given Euclidean structure $\mathcal{E}$ on $\mathbb{T}$ and a chart $\varphi : U \to \mathbb{E}^2$ for this structure, define a corresponding chart

$$\varphi \circ \zeta : \zeta^{-1}(U) \to \mathbb{E}^2.$$

All these charts form a new atlas for $\mathbb{T}$ and define a different Euclidean structure $\mathcal{E}\zeta$. Note that

$$\zeta : (\mathbb{T}, \mathcal{E}) \to (\mathbb{T}, \mathcal{E}\zeta)$$

is an equivalence of Euclidean structures. This action induces an action of $\mathrm{Homeo}(\mathbb{T})$ on $\mathrm{Sim}(\mathbb{T})$.

The quotient

- $\mathcal{M}_{\mathbb{T}} := \mathcal{S}(\mathbb{T})\,/\mathrm{Homeo}(\mathbb{T})$ is called the moduli space of $\mathbb{T}$ and the quotient

- $\mathcal{T}_{\mathbb{T}} := \mathcal{S}(\mathbb{T})\,/\mathrm{Homeo}_1(\mathbb{T})$ is called the Teichmüller space of $\mathbb{T}$. Note that there is a natural action of $M(\mathbb{T})$ on $\mathcal{T}_{\mathbb{T}}$ such that

$$\mathcal{M}_{\mathbb{T}} = \mathcal{T}_{\mathbb{T}}/M(\mathbb{T}).$$

Lemma 3.4.25. *A similarity σ of a complete metric X space with scale factor $\neq 1$ has a unique fixed point-*

Proof. Passing to σ^{-1} if necessary, we can assume that the scale factor is < 1. Then the sequence

$$\sigma^i(x)$$

is Cauchy. Its limit is a fixed point. Moreover, the fixed point is unique since the distance between two fixed points cannot shrink under σ. **q.e.d.**

Lemma 3.4.26. *Let M be a connected $(\mathcal{I},\mathcal{X})$-manifold. Let $\mathcal{H}$ be a subgroup of $\mathcal{I}$. Then the $(\mathcal{I},\mathcal{X})$-structure on M contains an $(\mathcal{H},\mathcal{X})$-structure for M if and only if the image of a holonomy $\eta : \pi_1(M) \to \mathcal{I}$ is contained in $\mathcal{H}$.*

Proof. The condition is clearly necessary.

So let us suppose that we have a holonomy $\eta : \pi_1(M) \to \mathcal{H}$ defined by some developing map $\nu : \tilde{M} \to \mathcal{X}$.

Let $\{\varphi_i : U_i \to \mathcal{X}\}$ be an $(\mathcal{I},\mathcal{X})$-atlas wherein each U_i is evenly covered. The sheets above the chart define a $(\mathcal{I},\mathcal{X})$-atlas for $\tilde{M}$.

Now, define new charts

$$\psi_i : U_i \to \mathcal{X}$$

by

$$\psi_i := \nu \circ \zeta_i$$

where $\zeta : U_i \to \tilde{M}$ is a homeomorphism identifying U_i with a sheet above it.

Let us consider a coordinate change:

$$\psi_i \circ \psi_j^{-1} = \nu \circ \zeta_i \circ \zeta_j^{-1} \circ \nu^{-1}.$$

Since $\zeta_i \circ \zeta_j^{-1}$ is a covering transformation, this coordinate change is locally $\nu \circ \tau \circ \nu^{-1}$ which is an element of H. Thus, $\{\psi_i : U_i \to \mathcal{X}\}$ is an $(\mathcal{H}, \mathcal{X})$-atlas for M. It is easy to check that it is compatible with the given $(\mathcal{I}, \mathcal{X})$-structure. **q.e.d.**

Proposition 3.4.27. *The set $\mathcal{S}(\mathbb{T})$ can be identified with the set of all $(\mathrm{Sim}(\mathbb{E}^2), \mathbb{E}^2)$-structures on $\mathbb{T}$.*

Proof. Every Euclidean structure on $\mathbb{T}$ induces a $(\mathrm{Sim}(\mathbb{E}^2), \mathbb{E}^2)$-structure in an obvious way. Moreover, two Euclidean structures that represent the same class in $\mathcal{S}(\mathbb{T})$ clearly define identical $(\mathrm{Sim}(\mathbb{E}^2), \mathbb{E}^2)$-structures. Thus, it remains to prove that every $(\mathrm{Sim}(\mathbb{E}^2), \mathbb{E}^2)$-structure is induced by a Euclidean structure on $\mathbb{T}$.

We claim that the maximal atlas for any $(\mathrm{Sim}(\mathbb{E}^2), \mathbb{E}^2)$-structure contains a subatlas that defines a Euclidean structure. By (3.4.26), we have to show that a holonomy takes values in $\mathrm{Isom}(\mathbb{E}^2)$.

Let $\nu : \tilde{\mathbb{T}} \to \mathbb{E}^2$ be any $(\mathrm{Sim}(\mathbb{E}^2), \mathbb{E}^2)$-map. We turn it into an isometry by pulling back the metric from $\mathbb{E}^2$. So let

$$\tau : \tilde{\mathbb{T}} \to \tilde{\mathbb{T}}$$

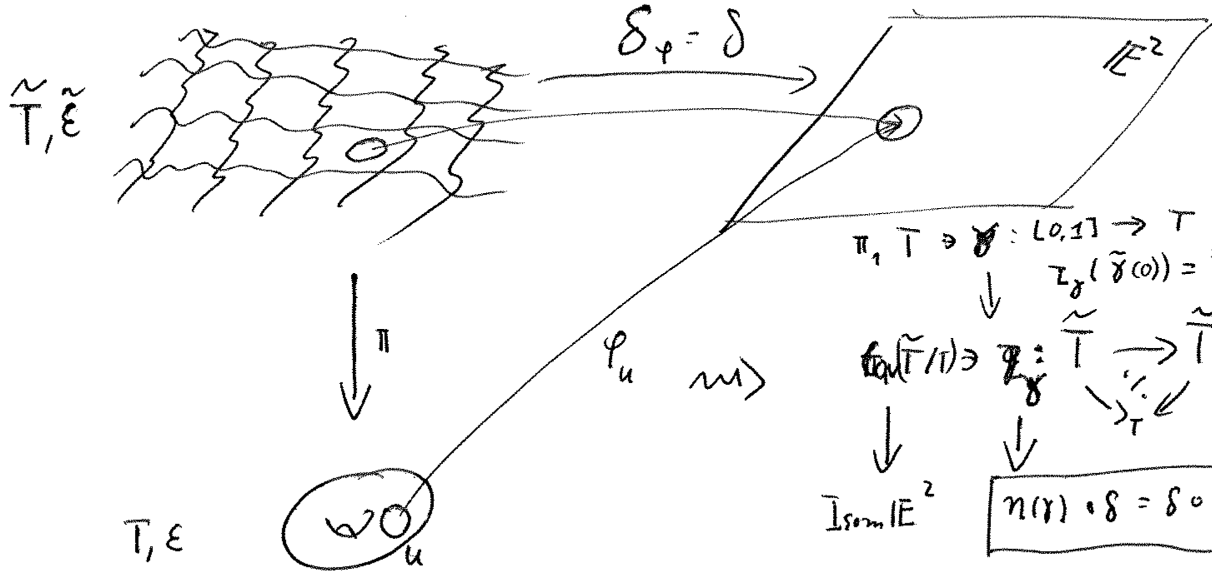
be a deck transformation. Thus, τ is a similarity in local coordinates. Since $\tilde{\mathbb{T}}$ is connected, all the scale factors agree. Using the triangle inequality for τ and τ^{-1}, we see that τ is a similarity of $\tilde{M}$. If τ is non-trivial, it has no fixed points and must be an isometry, being a similarity already.

The holonomy defined by ν sends a deck transformation τ to the similarity $\sigma : \mathbb{E}^2 \to \mathbb{E}^2$ that satisfies

$$\nu \circ \tau = \sigma \nu.$$

However, ν and τ are local isometries. Hence, so is σ. **q.e.d.**

$$\mathcal{J}(T) = {}_{\mathrm{Sim}\,\mathbb{E}^2}\backslash {}^{\mathcal{E}(T)}/_{\mathrm{Homeo}_1(T)}$$



$\varphi_u \rightsquigarrow$

$\pi_1 T \ni \gamma : [0,1] \to T$ (lift)

$\tau_\gamma(\tilde\gamma(0)) = \tilde\gamma(1)$

$$\begin{array}{ccc} \mathrm{Aut}(\tilde T/T) \ni & \tau_\gamma : & \tilde T \longrightarrow \tilde T \ \text{(over } T) \\ \downarrow & \downarrow & \\ \mathrm{Isom}\,\mathbb{E}^2 & \boxed{\eta(\gamma)\circ\delta = \delta\circ\tau_\gamma} & \end{array}$$

$$\Gamma_{\varepsilon,\delta} := \eta(\pi_1 T) \le \mathrm{Isom}\,\mathbb{E}^2$$

Thm 1

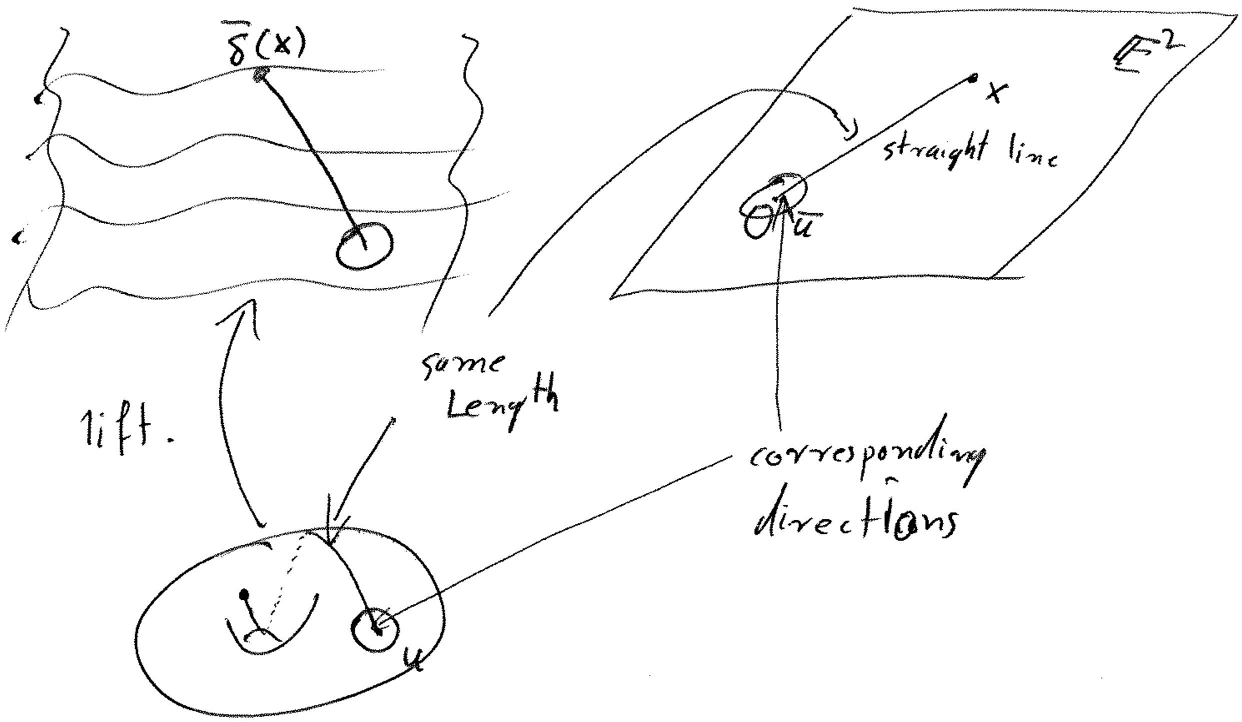
(a) $\delta : \tilde T \to \mathbb{E}^2$ is a homeomorphism.

(b) η is injective and the image is discrete.

(c) δ descends to a ~~homeom~~ geometric homeomorphism $T = \tilde T/_{\pi_1 T} \longrightarrow \mathbb{E}^2/_{\Gamma_{\varepsilon,\delta}}$

pf (outline)

Main difficulty is (a). construct an inverse $\bar\delta$ as follows:



easy: 1) $\bar{\delta}\,\delta = id_{\tilde{T}}$ $\qquad \delta\,\bar{\delta} = id_{\mathbb{E}^2}$

2) δ is local homeomorphism (δ is geometric!)

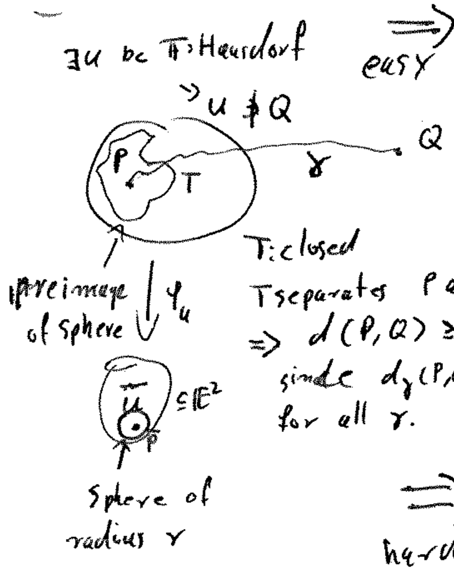
$\Rightarrow$ δ : continuous and open

$(1,2) \Rightarrow$ δ is a homeomorphism.

hard $\bar{\delta}$ exists

It is not clear that we can extend the initial geodesic segment in U to the required length.

Slogan : compactness



T: closed
T separates P and Q
$\Rightarrow d(P,Q) \geq r$
since $d_\gamma(P,Q) \geq r$ for all γ. $\rightarrow$

$\underset{\text{easy}}{\Longrightarrow}$ T, ε is metrically complete

ΓT has a metric by rectifiable paths and their lengths

(easy : pseudometric
hard : metric) ~~compact~~ $\lrcorner$

$\underset{\text{hard}}{\Longrightarrow}$ T, ε is geodesically complete.

Given that δ is a homeomorphism, the rest is easy :

δ is a $\pi_1 T$ - equivariant isometry

$\pi_1 T \curvearrowright \tilde{T}$ as $\mathrm{cov}(\tilde{T}/T)$

$\pi_1 T \curvearrowright \mathbb{E}^2$ via $\eta : \pi_1 T \rightarrow \mathrm{Isom}\, \mathbb{E}^2$

$\Rightarrow$, η injective ($\pi_1 T$ acts freely on $\tilde{T} \underset{\pi_1 T}{\cong} \mathbb{E}^2$)

, $\mathrm{im}\, \eta$ discrete (π_1 acts top. freely on $\tilde{T}$).

$\#$. δ descends to $\tilde{T}/\pi_1 T \longrightarrow \mathbb{E}^2/\pi_1 T$

and this is a homeomorphism . $\square$

$$\eta : \pi_1 T \longrightarrow \mathrm{Isom}\,\mathbb{E}^2$$

is a faithful, discrete representation.

$$\mathrm{Hom}^{f,d}(\pi_1 T, \mathrm{Isom}\,\mathbb{E}^2) := \{\text{faithful discrete rep.}\}$$

<u>Rem</u> η depends on δ.

$$\begin{tikzcd} \widetilde{T} \arrow[r, bend left, "\delta_1"] \arrow[r, bend right, "\delta_2"'] & \mathbb{E}^2 \end{tikzcd} \rightsquigarrow \exists\, \sigma_{12} \in \mathrm{Isom}\,\mathbb{E}^2 \leq \mathrm{Sim}\,\mathbb{E}^2 :$$

$$\begin{tikzcd} & \mathrm{Isom}\,\mathbb{E} \arrow[d, "\sigma_{12}"] \\ \pi_1 T \arrow[ur, "\eta_1"] \arrow[r, "\eta_2"'] \arrow[ur, phantom, shift right=3, "\circlearrowleft"] & \mathrm{Isom}\,\mathbb{E} \end{tikzcd}$$

$\Rightarrow$ $[[\eta]] \in \mathrm{Isom}\,\mathbb{E}^2 \backslash \mathrm{Hom}^{f,d}(\pi_1 T, \mathrm{Isom}\,\mathbb{E}^2)$

and

$[\eta] \in \mathrm{Sim}\,\mathbb{E}^2 \backslash \mathrm{Hom}^{f,d}(\pi_1 T, \mathrm{Isom}\,\mathbb{E}^2)$

are well defined.

$$\mathcal{D}(T) := \mathrm{Sim}\,\mathbb{E}^2 \backslash \mathrm{Hom}^{f,d}(\pi_1 T, \mathrm{Isom}\,\mathbb{E}^2)$$

is called <u>deformation space</u>.

We have

$$[-] : \mathcal{E}(T) \to \mathcal{D}(T)$$

<u>Thm</u> $[-]$ descends to a well defined bijection $\mathcal{T}(T) = {}_{\mathrm{Sim}\,\mathbb{E}^2}\backslash \mathcal{E}(T) / _{\mathrm{Homeo}_1 T} \to \mathcal{D}(T)$.

<u>pf</u> (a) well defined
(b) injective
(c) surjective

} all of these rely on the Dehn-Nielsen Thm.

<u>Thm</u> (Dehn-Nielsen)
S : closed surface, $\chi(S) \leq 0$. Then
$\nu : \mathrm{Map\,Cl}\; S \to \mathrm{Out}(\pi_1 S)$ is an isomorphism.

<u>Def of ν</u>

$h : S \to S$ homeo $\quad Q \in S$

$p : [0,1] \to S \quad p(0) = Q \quad p(1) = h(Q)$

$$\rightsquigarrow \quad \varphi_{h,p} : \pi_1(S, Q) \to \pi_1(S, Q)$$
$$[\gamma] \to [p + h\gamma + \overline{p}]$$

Let $p' : [0,1] \to S$ be a different path from Q to $h(Q)$.

$$\varphi_{h,p} = (p+\bar{p}') \, \varphi_{h,p'} \, \overline{(p+\bar{p}')}$$

Thus $\varphi_{h,p}$ and $\varphi_{h,p'}$ are conjugate by an element in $\pi_1(S, Q)$.

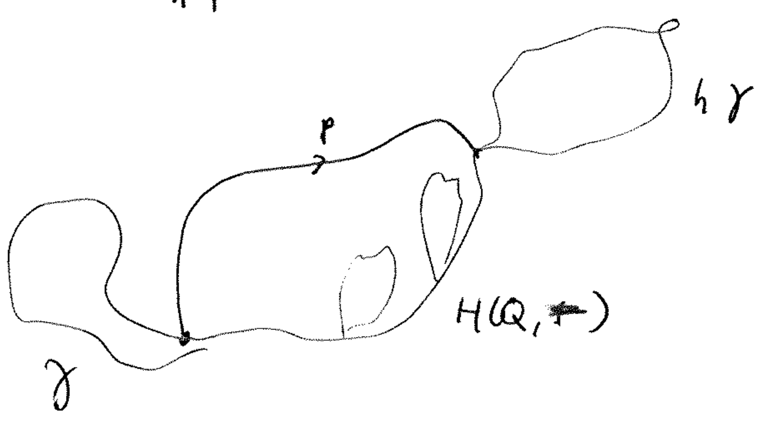
Suppose $h \simeq id$. Then there is a <u>good</u> path from Q to $h(Q)$:

$H : S \times I \to S$ $\quad h = H(-,1)$, $id_S = H(-,0)$

$\rightsquigarrow$ consider $H(Q, -) : [0,1] \to S$

Then $\varphi_{h,p}$ is <u>inner</u> indeed

$$\varphi_{h\,p} : [\gamma] \longmapsto (H(Q,\cancel{1}) + \bar{p}) \, \gamma \, (\cdots)$$



$\rightsquigarrow \nu : \text{Homeo}(S) / \text{Hom}$

$$\mathrm{Homeo}_1 S \longrightarrow \mathrm{Inn}(\pi_1 S)$$

$$\mathrm{Homeo}\, S \longrightarrow \mathrm{Out}(\pi_1 S)$$

$$\boxed{\begin{array}{l} \mathrm{Homeo}\, S / \mathrm{Homeo}_1 S \xrightarrow{\nu} \mathrm{Out}(\pi_1 S) \\ \quad \parallel \\ \mathrm{Map\,Cl} \end{array}}$$