Math 661 – Geometric Topology (homework 8, due Nov 02)

Definition. Let X and Y be topological spaces. A map $f : X \to Y$ is a homotopy equivalence if there is a map $h : Y \to X$ such that $h \circ f : X \to X$ is homotopic to the identity on X and $f \circ h : Y \to Y$ is homotopic to the identity on Y. (I.e., a homotopy equivalence is a map that induces an isomorphism in the homotopy category :-)

Two spaces X and Y are called homotopy equivalent if there is a homotopy equivalence $f : X \to Y$.

Note that any homotopy equivalences $f : X \to Y$ induces an isomorphism

$$f_* : \pi_1(X, \underline{x}) \to \pi_1(Y, f(\underline{x}))\,.$$

In particular, any self homotopy equivalence $f : X \to X$ induces an outer automorphism $\nu(f)$ of $\pi_1(X)$ in the same way that a self homeomorphism does.

Exercise 8.1. Let $f : T \to T$ be a self homotopy equivalence of the torus. Assume $\nu(f)$ is the class of inner automorphisms of $\pi_1(T)$. (This is, given a path p from $\underline{t}$ to $f(\underline{t})$, the homomorphism $p_* \circ f_*$ is an inner automorphism of $\pi_1(T, \underline{t})$.) Prove that f is homotopic to the identity.

Exercise 8.2. Let $f : T \to T$ be a self homotopy equivalence of the torus. Show that f is homotopic to a homeomorphism.

Exercise 8.3. Construct an example of a self homotopy equivalence of a finite graph Γ that is not homotopic to a homeomorphism.

Exercise 8.4. Let Γ be a finite graph. Let $f_0, f_1 : \Gamma \to \Gamma$ be two self homotopy equivalences of Γ. Prove that f_0 is homotopic to f_1 if $\nu(f_0) = \nu(f_1)$.

Exercise 8.5. Let R_n be the graph with one vertex v and n loops attached to the vertex. Show that every automorphism of $F_n := \pi_1(R_n, v)$ arises as an f_* for some homotopy equivalence

$$f : R_n \to R_n.$$

Exercise 8.6. Let T_1 and T_2 be two tori, and let $\phi : \mathrm{Cov}\left(\tilde{T}_1/T_1\right) \to \mathrm{Cov}\left(\tilde{T}_1/T_2\right)$ be an isomorphism. Show that there exists a homeomorphism $\tilde{\zeta} : \tilde{T}_1 \to \tilde{T}_2$ of the universal covers such that $\tilde{\zeta} \circ \tau = \phi(\tau) \circ \tilde{\zeta}$ holds for each deck transformation $\tau \in \mathrm{Cov}\,(T_1)$,

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.