We can easily define a homeomorphism on the 1-skeleton of these pictures that matches corresponding arcs. This homeomorphism extends to the 2-skeleta since any homeomorphism between the boundaries of two discs extends to their interiors. Thus, we have a homeomorphism

$$\tilde{C}_0 \cup \bigcup_i^j \tilde{C}_i^{(j)} \to C_0 \cup \bigcup_i^j C_i^{(j)}.$$

It remains to see that this homeomorphism extend continuously to the boundary. This, however, follows from the fact that an "ith-generation" disc is small and close to the boundary: The discs $\tilde{C}_i^{(j)}$ are small since their number per annulus at least doubles each step. The disc $C_i^{(j)}$ in the image is small by construction: It is within $\frac{1}{i}$-distance from its corresponding arc. This arc, in turn has diameter $\leq \frac{1}{i}$. **q.e.d.**

2.5 Consequences

Corollary 2.59. *Let S be a simple closed curve in $\mathbb{S}^2$. Then S cuts $\mathbb{S}^2$ into two discs intersecting in S.*

Corollary 2.60. *Let S be a simple closed curve in $\mathbb{S}^2$. Then any homeomorphism $\varphi : S \to \mathbb{S}^1 \subset \mathbb{S}^2$ extends to a homeomorphism $\mathbb{S}^2 \to \mathbb{S}^2$.*

Proof. This is clear since any homeomorphism of the boundaries of two discs extends to the interiors. The result follows from applying this to the two discs in $\mathbb{S}^2 - S$. **q.e.d.**

Corollary 2.61 (Schönflies Theorem, second form). *Let S be a simple closed curve in $\mathbb{E}^2$. Then every homeomorphism $S \to \mathbb{S}^1 \subset \mathbb{E}^2$ extends to a homeomorphism $\mathbb{E}^2 \to \mathbb{E}^2$.*

3 Manifolds

Definition 3.1. A (topological) m-manifold is a second countable Hausdorff space M wherein each point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^m$.

A chart is a pair $(U, \varphi : U \to \tilde{U})$ where U and $\tilde{U}$ are connected open subset of M and $\mathbb{R}^m$ respectively and where φ is a homeomorphism. A collection of charts whose domains cover M is called an atlas.

Let $(U_0, \varphi_0 : U_0 \to \mathbb{R}^m)$ and $(U_1, \varphi_1 : U_0 \to \mathbb{R}^m)$ be two charts. Put $V := U_0 \cap U_1$. Then the map

$$\begin{aligned} \xi : \varphi_0(V) &\to \varphi_1(V) \\ x &\mapsto \varphi_1\left(\varphi_0^{-1}(x)\right) \end{aligned}$$

is a homeomorphism called change of coordinates.

3.1 Euler Characteristic

Definition 3.2. An abstract simplicial complex is a set $\mathcal{V}$ of vertices together with a collection $\mathcal{S}$ of non-empty finite subsets called simplices containing all singleton subsets of $\mathcal{V}$ and satisfying the condition that any non-empty subset of a simplex is also a simplex. A simplicial map between abstract simplicial complexes is a map between their vertex sets such that the image of any simplex is a simplex in the target complex.

The realization of a simplicial complex $K = (\mathcal{V}, \mathcal{S})$ is defined as

$$|K| := \bigcup_{\sigma \in \mathcal{S}} |\sigma|$$

where $|\sigma|$ is the convex hull of the vertices in σ in $\mathbb{R}^{\mathcal{V}}$. Since $|K|$ is defined as a union, we will endow it with the weak topology. If you do not know what that is, never mind: we will consider finite simplicial complexes only, and for these, the weak topology

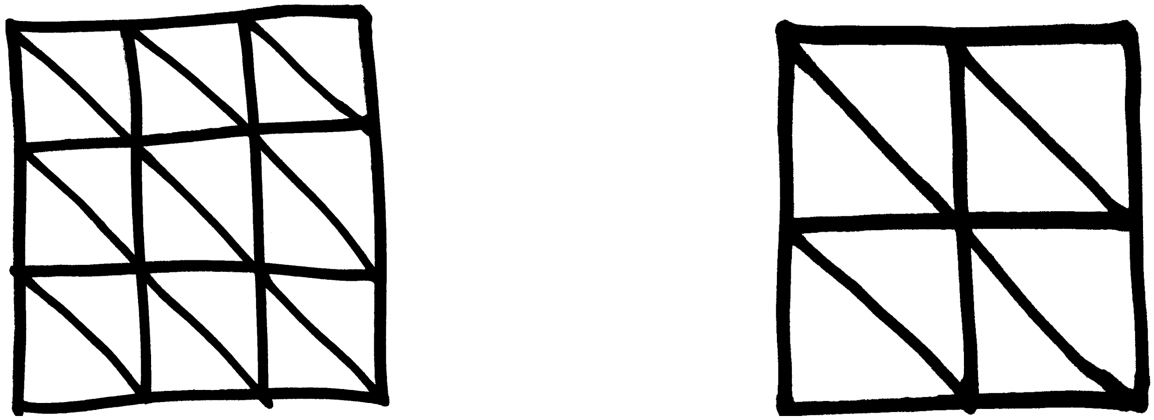


Table 10: Not everything that has triangles is a triangulation.

coincided with the subspace topology inherited from $\mathbb{R}^\nu$. Note that every simplicial map induces a continuous map between realizations.

A simplicial complex is the realization of an abstract simplicial complex sometimes with and sometimes without added information on what the vertices and simplices are.

Definition 3.3. Let X be a topological space. A triangulation of X is a simplicial complex that is homeomorphic to X. Sometimes, we will call the homeomorphism the triangulation.

Example 3.4. Figure (10) shows a triangulation and a non-triangulation of the torus.

Exercise 3.5. Find a triangulation of the torus that uses as few triangles as possible.

Definition 3.6. The closed star of a simplex τ in a simplicial complex K is the subcomplex of all simplices containing τ. The link of τ is the boundary of the star. Equivalently, it is the subcomplex of all those simplices σ such that $\sigma \cap \tau = \emptyset$ although $\sigma \cup \tau$ is a simplex.

There is an obvious way for a simplicial complex to be a manifold: All simplex links are spheres of the appropriate dimension. Those complexes are called <u>combinatorial manifolds</u>. In dimension 2, this is the only possibility.

Theorem 3.7. *Let K be a triangulated 2-manifold. Then K is a combinatorial 2-manifold, i.e., the link of each vertex is a subdivided circle.*

Proof. Every point in K has a neighborhood homeomorphic to an open disc. Hence, there are no isolated vertices. Moreover, every edge borders at least one triangle: Otherwise an interior point of that edge would separate every sufficiently small neighborhood, which is impossible in a 2-manifold as it does not happen in the plane. Similarly, the fact that no semicircle can separate the plane implies that every edge is, in fact, contained in at least two triangles.

Now we show that each edge is contained in at most two triangles. So suppose the edge e was in the intersection of at least three triangles. Then a point in the interior of e has a circle around it passing through two of these triangles. But that circle does not separate since you can bypass it along the third triangle. This contradicts the Jordan curve theorem which should hold near every point.

It follows that the link of every vertex is a disjoint union of circles. Since no point in a manifold can separate its neighborhoods, the link consists of one circle only. **q.e.d.**

Remark 3.8. In higher dimension, all sorts of bad things happen. There are manifolds that do not admit a combinatorial triangulation although they have a triangulation. In particular, that some links in a simplicial complex are not spheres does not imply that the complex is not a manifold. It is really hard to think how a vertex with a non-sphere link can have a neighborhood that is an open disc.

Definition 3.9. Let K be an abstract simplicial complex. A <u>subdivision</u> of K is an abstract simplicial complex L such that

1. The vertices of L are points in $|K|$.
2. Every simplex of L is contained in the realization of a simplex of K.
3. The induced linear map $|L| \to |K|$ is a homeomorphism.

Two simplicial complexes are called combinatorially equivalent if they have isomorphic subdivisions.

Example 3.10. Any two subdivisions of a given simplicial complex K are combinatorially equivalent. In fact, they have a common refinement.

Proof. Since all simplices of the subdivisions are contained in simplices of K and look straight therein, their intersections, if non-empty, are convex. From here, a common finer subdivision is easily found. **q.e.d.**

Fact 3.11. *Every compact 2-manifold Σ has a triangulation and any two triangulations of Σ are combinatorially equivalent.*

Remark 3.12. In higher dimensions, it is not true that all manifolds have triangulations, and there are manifolds that admit combinatorially inequivalent triangulations. Even when we restrict ourselves to combinatorial triangulations to begin with, there are inequivalent ones. !!! give a reference !!!

Corollary 3.13. *Any invariant of 2-manifolds defined in terms of combinatorial equivalence classes of triangulations is, in fact, a topological invariant of the manifold.*

Example 3.14. The Euler characteristic of a simplicial complex is the alternating sum of the numbers of simplices in different dimensions, i.e.,

$$\chi(K) := \sum_{m \geq 0} (-1)^m \left| \{ \sigma \in K \mid \dim(\sigma) = m \} \right| .$$

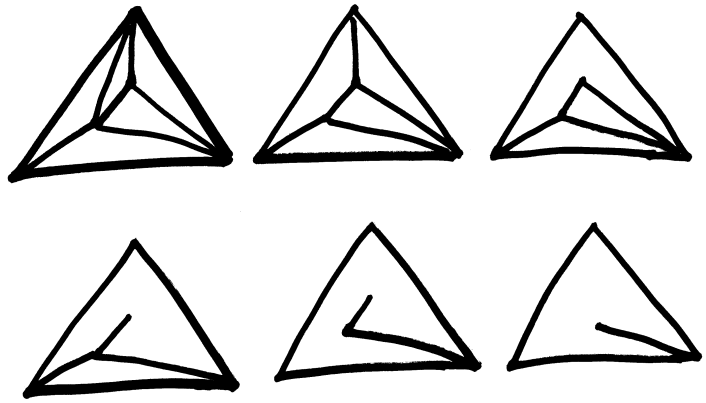


Table 11: The deletion proof

If L is a subdivision of K, then $\chi(L) = \chi(K)$. Hence, the Euler characteristic of a surface is a topological invariant.

Proof. In dimension 2 the "deletion proof" works: Inside the triangles, delete edges one by one decreasing the number of regions and edges by one. If there is only one region left, delete interior vertices along with their edges (push in free faces!). Finally, delete vertices in the 1-skeleton. See figure (11)

Warning: This proof does not work in higher dimensions. Removing the 1-dimensional material is possible only because we can find terminal vertices. In dimension 3, we would be left with the task of removing 2-complexes. However, we might run into something like Bing's house (see figure 12) where we do not find any "free faces" to push in. **q.e.d.**

Exercise 3.15. Give a proof for the invariance of the Euler-characteristic with respect to subdivisions that works in all dimensions.

Remark 3.16. Since the Euler characteristic can also be computed from the rank of singular homology groups, it turns out, that the Euler characteristic is a topological invariant for all triangulable spaces, i.e., any two triangulations of the same space have the same

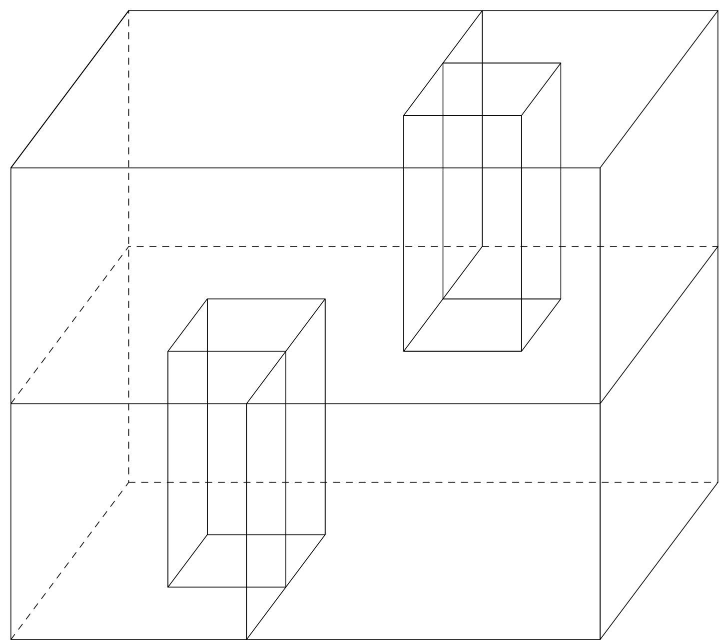


Table 12: Bing's house

Euler characteristic, even if they are not combinatorially equivalent.

3.2 Triangulability of Surfaces

Theorem 3.17 (Rado 1925). *Every 2-manifold Σ has a triangulation.*

We present Rado's original proof since it is efficient and not technical.

Proof of theorem 3.17. We need a little bit of terminology. Call an embedded closed disc J in Σ a notionJordan domain if it is contained in a chart. Since the topology of Σ has a countable basis, Σ is covered by countably many charts. Each chart, in turn, allows for a countable set of Jordan domains whose interiors cover the chart. Thus, we find a sequence $J_1, J_2, \ldots$ of Jordan whose interiors cover Σ.

Claim A. There is a sequence $J_1^*, J_2^*, \ldots$ of Jordan domains whose interiors cover Σ such that $\partial\left(J_i^*\right) \cap \partial\left(J_j^*\right)$ is finite for $i \neq j$.

PROOF. Put $J_1^* := J_1$. Now suppose that $J_1^*, J_2^*, \ldots, J_r^*$ have already been constructed such that:

1. The regions J_i^* "thicken" the original domains J_i, i.e., we have $J_i \subseteq J_i^*$ for $i = 1, 2, \ldots, r$.
2. The set of intersections

$$M_r := \bigcup_{i<j\leq r} \partial\left(J_i^*\right) \cap \partial\left(J_j^*\right)$$

   is finite.

Our task will be to find the next term J_{r+1}^* such that the above two conditions are preserved.

Some more local definition will ease the argument. We call the points in M *crossings*, and a path in Σ is *admissible* if it intersects the set

$$B := \bigcup_{i\leq r} \partial\left(J_i^*\right)$$

in only finitely many points. Given an open set $U \subseteq \Sigma$, call two points *U-equivalent* if they can be connected by an admissible path in U. Note that U-equivalence is an equivalence relation.

Let P be a point outside M_r. Any neighborhood U of P contains a sub-neighborhood V such that any two point in V are U-equivalent. Indeed, if P does not lie in B, we can choose V so that it is connected and does not intersect B. In this case, any two points in V are even V-equivalent. If $P \in B - M_r$ then we chose V to intersect only one of the boundary curves $\partial\left(J_i^*\right)$. The Schönflies Theorem implies that we can pretend this curve is the unit circle in the plain. In this picture, however, the claim is obvious.

It follows that for any open set U that does not contain any crossings, the U-equivalence classes are open. Thus, they coincide with the components of U. In particular, if U is connected, any two points in U can be connected by an admissible path.

Now, we can proceed to construct J^*_{r+1}. Consider a chart that contains J_{r+1}. By the Schönflies Theorem we can assume that J_{r+1} is represented as the closed unit disc in this chart. Let R be an open annulus around J_{r+1} that does not contain any crossings. Thus, R intersects B in a union of disjoint arcs. The set $R - B$ is non-empty and open. Thus there are two points $P, Q \in R$ such that:

1. The two radii from the center of J^*_{r+1} through P and Q do not intersect and hence separate the annulus into two topological rectangles R_0 and R_1.
2. Both points have open neighborhoods in $R - B$. Thus, we can find points P_0, P_1 close to P and points Q_0, Q_1 close to Q such that the segments PP_i and QQ_i are contained in the rectangle R_i and do not intersect B.

Since the rectangles are connected and do not contain any crossings, we know that P_i and Q_i can be connected inside R_i by a path p_i that intersects B only finitely many times.

Now we concatenate: the path

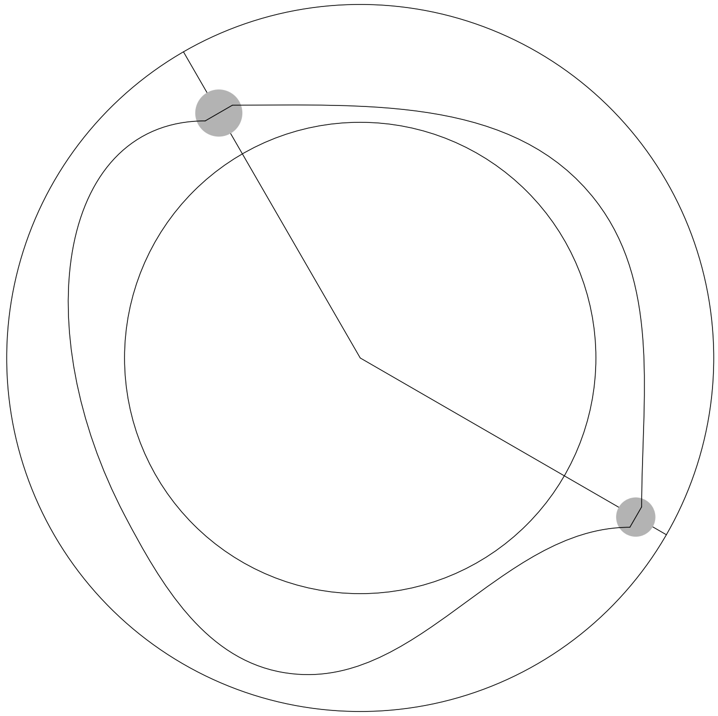
$$(QP_0)\, p_0\, (P_0P)\, (PP_1)\, p_1\, (Q_1Q)$$

is a closed loop inside R surrounding J_{r+1}. Deleting pieces if necessary to avoid self-intersection, we find a simple closed curve inside R that intersects B only finitely many times and whose interior contains J_{r+1}. This is our choice for J^*_{r+1}. □

Now, we define a sequence

$$P_1^{(1)}, P_2^{(1)}, \dots, P_2^{(s_2)}, P_3^{(1)}, \dots, P_3^{(s_3)}, \dots$$

of closed discs that cover Σ such that the following hold:



!!! finish this !!!

Table 13: The construction of J^*_{r+1}

1. For any r, we have $\bigcup_{i\le r} J^*_i = \bigcup_{i\le r}\bigcup_{j=1}^{s_i} P_i^{(j)}$.
2. The interiors of the $P_i^{(j)}$ are pairwise disjoint.
3. Each point is contained in only finitely many $P_i^{(j)}$. Note that by compactness of closed discs, this condition implies that any of these discs meets only finitely many other discs of the sequence.

Thus, we can think of these $P_i^{(j)}$ as a polygonal decomposition of Σ which is easily turned into an honest triangulation.

Hence, we are reduced to proving the existence of the sequence $P_1^{(1)}, P_2^{(1)}, \ldots, P_2^{(s_2)}, P_3^{(1)}, \ldots, P_3^{(s_3)}, \ldots$. Put $P_1^{(1)} := J^*_1$. Suppose already have constructed

$$P_1^{(1)}, P_2^{(1)}, \ldots, P_2^{(s_2)}, P_3^{(1)}, \ldots, P_3^{(s_3)}, \ldots P_r^{(1)}, \ldots, P_r^{(s_r)}.$$

The Jordan domain J^*_{r+1} is chopped up into regions by the boundary curves $\partial(J^*_i)$ for $i \leq r$. Some of these regions might not be discs but contain finitely many holes. We further subdivide and arrive at a decomposition of J^*_{r+1} into finitely many discs. Among these we chose as $P^{(1)}_{r+1}, \ldots, P^{(s_r+1)}_{r+1}$ precisely those that do not contain any interior point of $\bigcup_{i \leq r} J^*_i$.

Of the three requirements our sequence is supposed to meet, only (3) requires proof. So let P be a point in Σ. There is a Jordan domain J^*_k containing P as an interior point. Let U be a neighborhood of P in J^*_k. A disc $P_i(j)$ can intersect U only if $i \leq k$. This establishes (3) and completes the proof. **q.e.d.**

3.3 Geometric Structures

Definition 3.18. A <u>differentiable structure</u> on a manifold is an atlas maximal with respect to the restriction that all coordinate changes are differentiable maps.

A <u>complex structure</u> on a manifold is an atlas maximal with respect to the restriction that all coordinate changes are holomorphic maps. A map $\xi : \mathbb{R}^2 \to \mathbb{R}^2$ is <u>holomorphic</u> if it is differentiable its derivative is a matrix of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

A <u>Euclidean structure</u> on a manifold is an atlas maximal with respect to the restriction that all coordinate changes are Euclidean isometries.

Given a fixed homeomorphic identification of $\mathbb{R}^m$ with hyperbolic m-space, we can define a <u>hyperbolic structure</u> on a manifold as an atlas maximal with respect to the restriction that all coordinate changes are hyperbolic isometries.

Example 3.19. There is no Euclidean structure on the sphere $\mathbb{S}^2$.