

3.5.3 Classification of Homeomorphisms

Let us consider orientation preserving homeomorphisms of the torus T up to homotopy. They form the group

$$\mathrm{SL}_2(\mathbb{Z}).$$

A matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ can be studied by looking at its trace.

Definition 3.5.18. M is elliptic if $|\mathrm{tr}(M)| < 2$.

M is parabolic if $|\mathrm{tr}(M)| = 2$.

M is hyperbolic if $|\mathrm{tr}(M)| > 2$.

The significance lies in the fact that the characteristic polynomial of M is given by:

$$\mathrm{Det}(M) - \lambda \mathrm{tr}(M) + \lambda^2.$$

Thus we have:

M is elliptic: In this case, we have two complex conjugate eigenvalues λ_1, λ_2 . Thus there is a fixed point in Teichmüller space. There are only three possibilities:

$\mathrm{tr}(M) = 0$: We find $\lambda_1 = i$ and $\lambda_2 = -i$. Thus, the matrix has order four. This homeomorphism is the rotation by $\frac{\pi}{2}$. The standard Euclidean structure (identify opposite edges of a unit square) is the fixed point in Teichmüller space.

$\mathrm{tr}(M) = -1$: Here λ_i is a sixth root of unity and M has order six. Here, we expect the homeomorphism to be a rotation by $\frac{\pi}{3}$. Moreover, the Euclidean structure should correspond to a shape of a fundamental domain. Thus, we represent the torus as a regular hexagon with opposite edges identified. A rotation around the center is our homeomorphism. It might take you some time to convince yourself that a hexagon really gives a torus when you identify opposite edges. You can see this, however, from the induced tessellation of the Euclidean plane by hexagons.

$\text{tr}(M) = 1$: Finally, we find λ_i is a third root of unity, and M has order three. This homeomorphism is the square of the previous one.

Thus, elliptic elements are periodic. They have finite order. Moreover, since one the eigenvalues lies in the upper half plane, there is a fixed point in Teichmüller space.

M is parabolic: Here $\lambda_1 = \lambda_2 = 1$. Consider the action on \mathbb{R}^2 . There is an eigenspace. Since the eigenvalue is 1, this line is fixed point wise. Will it descend to a closed curve on the torus $\mathbb{R}^2/\mathbb{Z}^2$?

We have to check if the slope of this line is rational. This follows from the fact that all entries in M are rational whence we can interpret the singularity of $M - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ over \mathbb{Q} . Thus, this homeomorphism fixes a closed curve on the torus.

Note that $M(T) = \text{SL}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$. Thus, all parabolic elements are conjugate to elements of the form

$$\pm \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Thus, parabolic elements are powers of Dehn twists.

M is hyperbolic: Now we have two real eigenvalues. Their product is $\text{Det}(M) = 1$. The eigenvalues are not rational. For suppose the eigenspaces had rational slope. Then they would descend to closed curves on the torus and in the universal cover these would be represented by a family of parallel lines. However, the contracting eigenvalue would have to shrink the lattice of intersections with its eigenspace. Hence the pattern upstairs can only be invariant if it is dense.

Thus, the invariant lines descend to bi-infinite geodesic curves on the torus. This is the most elementary example of a geodesic

lamination. Thus, a hyperbolic homeomorphism leaves invariant a pair of geodesic laminations.