C Finiteness Properties

Definition C.1. A group G is of <u>type F_m</u> if it has an Eilenberg-MacLane complex $\mathrm{K}(G,1)$ with finite skeleta in dimensions $\leq m$.

Remark C.2. We do not simply require the m-skeleton to be finite so that the definition makes sense for $m=\infty$ in which case we require the Eilenberg-MacLane complex to have finitely many cells in each dimension.

Example C.3. Free groups are of type F_∞. So is Thompson's group F. Grigorchuk's group is finitely generated but finitely presented.

Observation C.4. *Prove that G is of type F_m if and only if there is an $(m-1)$-connected simplicial complex X with a free and cocompact G-action.*

Corollary C.5. *Prove that every group is of type F_0, a group is finitely generated if and only if it is of type F_1, and it is finitely presented if and only if it is of type F_2.* **q.e.d.**

Exercise C.6. Let G be of type F_m and let X be a $(m-2)$-connected simplicial complex of dimension $m-1$ with a free, cocompact G-action. Show that X embeds G-equivariantly into a $(m-1)$-connected simplicial complex of dimension m with a free, cocompact G-action.

Infer that a group is of type F_∞ if and only if it is of type F_m for all $m<\infty$.

Exercise C.7. Let $N \hookrightarrow G \twoheadrightarrow Q$ be a short exact sequence of groups where N is of type F_{m-1} and G is of type F_m. Then Q is of type F_m.

Exercise C.8. Show that all finite groups are of type F_∞.

C.1 Brown's Criterion

The big lemma about finiteness properties is due to K.S. Brown [Bro87a]

Theorem C.9. *Let G be a group and D a directed set. Let $(X_\alpha)_{\alpha \in D}$ be a directed system of G-CW-complexes upon which G acts cocompactly by cell-permuting homeomorphism. Assume that the $\varinjlim_{\alpha \in D} X_\alpha$ is $(m-1)$-connected and that, for each $\alpha \in D$ and each cell p in the m-skeleton of X_α, the stabilizer $\mathrm{Stab}(p)$ is of type $\mathrm{F}_{m-\dim(p)}$. Then the following are equivalent:*

1. *G is of type F_m.*
2. *For each $i < m$, the directed system of homotopy groups $(\pi_i(X_\alpha))_{\alpha \in D}$ is essentially trivial.*

Here a directed system of group $(H_\alpha)_{\alpha \in D}$ is called essentially trivial if for each $\alpha \in D$ there is an element $\beta > \alpha$ such that the homomorphism $H_\alpha \to H_\beta$ is trivial.

Corollary C.10. *Suppose G act cocompactly by cell permuting homeomorphisms on an $(m-1)$-connected CW-complex X such that for every cell p, the stabilizer $\mathrm{Stab}(p)$ is of type $\mathrm{F}_{m-\dim(p)}$. Then G is of type F_m.*

Proof. Consider the directed system

$$X \xrightarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} \cdots$$

and check that it satisfies the hypotheses of Brown's Criterion. **q.e.d.**

C.2 Applications of Brown's Criterion

Example C.11. Let G be a group and let $D := \{K \subseteq G \mid K \text{ is finite }\}$ be the set of finite subsets of G directed by inclusion. For $K \in D$, define the simplicial complex

$$X_K := \{\sigma \mid \sigma \subseteq gK \text{ for some } g \in G\}.$$

Obviously, G acts cocompactly on X_K. Simplex stabilizers conjugate into K and are, therefore, finite. Thus G is of type F_m if and only if $(\pi_i(X_\alpha))_{\alpha\in D}$ is essentially trivial for all $i < m$.

Let H be a subgroup of finite index in G. Then the induced action of H on X_K is still cocompact. Thus we can use the same directed system to detect finiteness properties of H. Thus, we have

Corollary C.12. *Let H be a subgroup of finite index in G. Then H is of type F_m if and only if G is of type F_m.*

With only little more effort, the same construction yields:

Exercise C.13. Let G be of type F_m and let H be a retract of G. Then H is of type F_m.

Proposition C.14. *Let $N \hookrightarrow G \twoheadrightarrow Q$ be a short exact sequence of groups where N and Q are of type F_m. Then G is of type F_m.*

Proof. Take any free Q complex that proves Q to be of type F_m. Consider this complex as a G-complex where the G-action is given via the projection $G \to Q$. Then, all cell stabilizers equal N and are of type F_m. Thus the same complex proves that G is of type F_m. **q.e.d.**

Definition C.15. Let X and Y be two metric spaces. A map $f : X \to Y$ is called a lipschitz if there are constants L and K such that

$$d(x, y) \leq Ld(f(x), f(y)) + K$$

for all $x, y \in X$.

A lipschitz $f : X \to Y$ is a quasi-retraction if there exists a lipschitz $h : Y \to X$ and a constant C such that

$$d(f(h(y)), y) \leq C$$

for all $y \in Y$. The map h is called a quasi-section for f. If there is a quasi-retraction $f : X \to Y$, the space Y is called a quasi-retract of X.

The map f is a quasi-isometry if there is a map $h : Y \to X$ such that f is a quasi-retraction with quasi-section h and h is a quasi-retraction with section f.

Two finitely generated groups are called quasi-isometric if they have quasi-isometric Cayley graphs for some finite generating sets. The notion of a quasi-retract carries over to group in the same way.

Observation C.16. *Since compositions of lipschitz maps are lipschitz, quasi-isometry is an equivalence relation on the class of metric spaces.*

Exercise C.17. Show that X and Y are quasi-isometric if and only if there is a map $f : X \to Y$ and constants L, K, and C such that the following hold:

1. f is bilipschitz, i.e., $\frac{d(f(x),f(y))}{L} - K \leq d(x,y) \leq Ld(f(x), f(y)) + K$ for all $x, y \in X$.
2. f is quasi-surjective, i.e., $Y = \bigcup_{x \in X} \mathbb{B}_C(x)$.

Exercise C.18. Let Σ and Ξ be two finite generating sets for G. Prove that $\Gamma_\Sigma(G)$ and $\Gamma_\Xi(G)$ are quasi-isometric.

Proposition C.19 ([Alon94]). *Let G be of type F_m. If H is a quasi-retract of G, then H is of type F_m.*

Proof. Our directed set will be $\mathbb{N}$. For $n \in \mathbb{N}$, define

$$\begin{aligned} X_n &:= \{\sigma \subseteq G \mid \operatorname{diam}(\sigma) \leq n\} \\ Y_n &:= \{\sigma \subseteq H \mid \operatorname{diam}(\sigma) \leq n\} \end{aligned}$$

The group G acts on X_n cocompactly and with finite stabilizers. The same holds for H and Y_n. Moreover, both directed systems converge to big simplices. Thus, we can use these directed systems to determine the finiteness properties of these groups.

Let $f : G \to H$ be a retraction with quasi-section $h : H \to G$ with constants L, K and C as in the definition. Then, we form

$$Y_m \xrightarrow{h} X_M \to X_N \xrightarrow{f} Y_n$$

where $M \geq L(m+K)$ and N is chosen so that the middle map annihilate homotopy in dimensions $< m$. The number n, again, is derived from the lipschitz constants. Now the composite map is induced by $f \circ h$ which is homotopic to the inclusion map into $Y_{n'}$ for any $n' \geq n + 2C$. Here, we use that fact that two simplicial maps f and h are homotopic if, for each simplex σ, the union $f(\sigma) \cup h\sigma$ is a simplex.

Hence the inclusion $Y_m \subseteq Y_{n'}$ annihilates homotopy groups in dimensions $< m$. This implies that H is of type F_m. **q.e.d.**

Corollary C.20. *Finiteness properties are geometric, i.e., they depend only on the quasi-isometry type of a group.*

C.3 The Stallings-Bieri Series

Finiteness properties are not yet well understood. We have, however, some series of groups for which finiteness properties have been established. In this section, we shall discuss the most accessible example of such a series, which is due to J.R. Stallings and R. Bieri.

Consider the exact homomorphism

$$F_{\{x_1,y_1\}} * F_{\{x_2,y_2\}} * \cdots * F_{\{x_n,y_n\}} \to \mathbb{Z}$$

that sends all generators x_i, y_i to 1 and let

$$G_n$$

be the kernel.

Proposition C.21. *The group* G_n *is of type* F_{n-1} *but not of type* F_n.

Proof. The free group $F_{\{x_i,y_i\}}$ acts freely and cocompactly on a regular tree T_i all of whose vertices have degree 4. The homomorphism

$$\begin{aligned} F_{\{x_i,y_i\}} &\rightarrow \mathbb{Z} \\ x_i,\, y_i &\mapsto 1 \end{aligned}$$

induces an action of $F_{\{x_i,y_i\}}$ on $\mathbb{R}$ by translations. With respect to this action, we have a height function

$$h_i : T_i \to \mathbb{R}$$

that is $F_{\{x_i,y_i\}}$-equivariant. Note that at each vertex of T_i we have two ascending edges (labeled by the generators) and two descending edges (labeled by their inverses).

Put

$$X := \bigtimes_i T_i$$

and consider the height

$$\begin{aligned} h : X &\rightarrow \mathbb{R} \\ (t_i) &\mapsto \sum_i h_i\,(t) \end{aligned}$$

The ascending and descending links of vertices in X are spheres of dimension $n-1$. In fact, they arise as joins of ascending, respectively descending, links in the factors T_i. It follows by the Morse lemma (B.3) that slices

$$X_t := h^{-1}\,([-t,t])$$

are $(n-2)$-connected: As t increases, we cone off $(n-2)$-connected subcomplexes, thereby not changing homotopy groups in dimensions $\leq n-2$. However, in the limit, we obtain the contractible space X. Thus, the homotopy groups in dimensions $\leq n-2$ were trivial all along.

We use X_t as a directed system of CW-complexes to apply Brown's Criterion. It is obvious that G_n acts freely and cocompactly on any X_t. Since the complexes X_t are already $(n-2)$-connected, it follows that G_n is of type F_{n-1}.

As for the other direction, we have to prove that the directed system $(X_t)_t$ is not essentially $(n-1)$-connected. Note that X, being a product of trees, is a metric space with unique geodesics. Thus, for each vertex $v \in X$, we have a geodesic retraction

$$X - \{v\} \to \mathrm{Lk}\,(v)\,.$$

It follows that a sphere in X is not 0-homotopic in $X - \{v\}$ unless it has a 0-homotopic image in $\mathrm{Lk}\,(v)$. Moreover, we have a retraction

$$\mathrm{Lk}\,(v) = \mathop{\ast}_i \{x_i^{\pm}, y_i^{\pm}\} \to \mathbb{S}^{n-1} = \mathop{\ast}_i \{x_i, y_i\}$$

induced by $x_i^{\pm} \mapsto x_i$ and $y_i^{\pm} \mapsto y_i$. This way, we recognize ascending links of vertices as retracts of the links.

Now, we are ready to construct, for any specified number t, an $(n-1)$-sphere in X_0 that does not die in X_t. To do this, let us fix a vertex $v \in X$ whose height is $\leq t$. The ascending link $\mathrm{Lk}^{\uparrow}(v)$ is a sphere of dimension $n-1$. It is spanned by the ascending edges starting at v. Each of these edges lives in some component tree T_i and can be extended in this factor to a geodesic ray. This way, we can move the sphere up inside X until it reaches height 0. Let $\mathbb{S}$ be the sphere obtained that way. Since we used geodesic rays, $\mathbb{S}$ maps homeomorphically to $\mathrm{Lk}^{\uparrow}(v)$ under the geodesic retraction

$$X - \{v\} \to \mathrm{Lk}\,(v)$$

and is, by our previous considerations, not 0-homotopic in $X - \{v\} \supseteq X_t$.

It follows that the directed system X_t is not essentially $(n-1)$-connected whence G_n is not of type F_n. **q.e.d.**