Math 739 – Important Groups (homework 5, due Mar 1)

Exercise 5.1. Prove that any finitely generated group

(a) has only finitely many normal subgroups of index 2002.

(b) has only finitely many subgroups of index 2002.

Exercise 5.2. Show that finitely generated free groups are Hopfian: a group G is Hopfian if every surjective endomorphism $G \twoheadrightarrow G$ is an automorphism. Hint: One way to do this is to show that residually finite groups are Hopfian. If you want to take this route, then you might want to use the result of exercise (5.1).

Corollary. *Every minimal set of generators for F_n is a set of free generators.*

Proof. Let Σ be a minimal generating set for F_n. The inclusion $\Sigma \hookrightarrow F_n$ extends to a group homomorphism $F_\Sigma \to F_n$ which is onto because Σ generates F_n. On the other hand, by exercise (4.1), $|\Sigma| = \mathrm{rk}\,(F_n) = n$. Hence we have a surjection of free groups of the same rank. Since these groups are Hopfian, the map $F_\Sigma \to F_n$ is an isomorphism. **q.e.d.**

Exercise 5.3. Show that $F_2 \times C_\infty$ does not have the finite intersection property. That is, find two finitely generated subgroups whose intersection is not finitely generated.

Exercise 5.4. Find an *efficient* algorithm to solve the conjugacy problem in finitely generated free groups.

Exercise 5.5. Modify the graphs-and-folds technique used in proving Grushko's Theorem to devise an algorithm that does the following:

> The input it takes is a finite set $\{g_1, \ldots, g_r\}$ of elements in F_n given as reduced words in the standard generators.
>
> The output is a list of free generators $\{h_1, \ldots, h_s\}$ for the subgroup $\langle g_1, \ldots, g_r \rangle$ generated by the g_i.

Exercise 5.6 (extra credit). Find an algorithm that solves the subgroup membership problem for finitely generated free groups.

Exercise 5.7. Show that the automorphism φ of the infinite binary rooted tree defined by

$$\varphi = (1, \varphi)\sigma$$

has infinite order.

It suffices to solve, on the average, half of the problems correctly.