Math 739 – Important Groups (homework 12, due Apr 29)

Exercise 12.1. Show that $\mathrm{SL}_2(\mathbb{Z})$ contains a non-abelian free subgroup of finite index.

Exercise 12.2. Let G act on a tree T such that the following conditions are satisfied:

1. G acts transitively on the set of geometric (unoriented) edges.
2. G does not act transitively on the set of vertices.
3. T has no terminal vertices and is not isomorphic to a line.

Let e be an edge in T that connects the vertices v and w. Let G_v and G_w denote the stabilizers of these vertices and let G_e denote the stabilizer of the edge e. Then $G = G_v *_{G_e} G_w$.

Infer that

$$\mathrm{SL}_2(\mathbb{Z}) = C_6 *_{C_2} C_4.$$

Exercise 12.3. Prove that for some finite generating set and some constant C, geodesic paths in the Cayley graph of $\mathrm{SL}_2(\mathbb{Z})$ have the C-fellow traveler property. Hint: Use geodesic paths and the action on the tree.

Exercise 12.4. Let G be of type F_m and let X be a $(m-2)$-connected simplicial complex of dimension $m-1$ with a free, cocompact G-action. Show that X embeds G-equivariantly into a $(m-1)$-connected simplicial complex of dimension m with a free, cocompact G-action.

Infer that a group is of type F_∞ if and only if it is of type F_m for all $m < \infty$.

Exercise 12.5. Let $N \hookrightarrow G \twoheadrightarrow Q$ be a short exact sequence of groups where N is of type F_{m-1} and G is of type F_m. Then Q is of type F_m.

Exercise 12.6. Show that all finite groups are of type F_∞.

Exercise 12.7. Let G be of type F_m and let H be a retract of G. Then H is of type F_m.

Exercise 12.8. Show that every finitely generated nilpotent group is finitely presented. Extra Credit: What about higher finiteness properties?

It suffices to solve, on the average, half of the problems correctly.