where H runs through all finitely generated subgroups of G. Since any compact (finite) subset of G is contained in one of these subgroups, this representation has almost invariant vectors — just consider the action of the finite subset on an appropriate summand where it fixes the coset of the identity. Since G is supposed to be Kazhdan, we conclude that the representation has an invariant vector.

Hence one of the summands has an invariant vector. Such a vector corresponds to a constant function on G/H. Hence this quotient is finite. Therefore, G is virtually finitely generated and hence finitely generated. **q.e.d.**

Corollary 1.46. *Locally indicable groups do not have Kazhdan's property (T).*

Proof. Being discrete and Kazhdan, the group is finitely generated. Being finitely generated and locally indicable, it is indicable. **q.e.d.**

1.4 The Geometry of the Cayley Graph

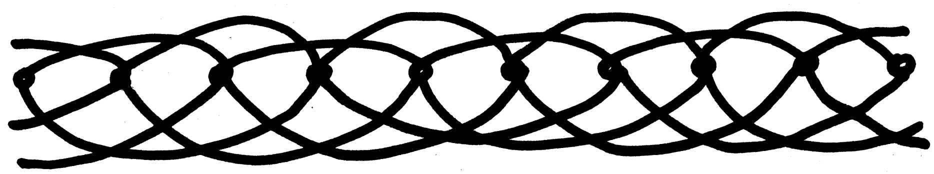
Definition 1.47. Let G be a group with finite generating system Σ. The (left) Cayley graph $\Gamma_\Sigma(G)$ is a (directed and labeled) graph. Its set of vertices is G, and for each vertex $g \in G$ and each generator $x \in \Sigma$, there an edge (labeled by x) from g to gx. Note that G acts from the left on $\Gamma_\Sigma(G)$.

There is a corresponding notion of a right Cayley graph upon which G acts from the right.

Remark 1.48. We usually do not care about the direction of edges or the labeling. Thus we regard the Cayley graph as a metric space: every edge has length 1 and the distance of any two points is the length of the shortest path connecting them. This length is finite since the Cayley graph is connected — this follows from the assumption that Σ generated G: any element of G can be written as a word in the generators (and their inverses) and this translates into a path connecting the group element to the identity element.

Example 1.49. Here is the Cayley graph $\Gamma_{\{1\}}(C_\infty)$:

And here this is what $\Gamma_{\{2,3\}}(C_\infty)$ looks like:



Let us provide one example of how to make use of the Cayley graph. We already now ba (1.44) that a group is finitely generated if it has a finitely generated subgroup of finite index. As we shall see, the converse holds true, as well. This follows from the following lemma when applied to actions on Cayley graphs.

Lemma 1.50. *Let the group G act on the connected topological space X and suppose that there is an open subset U such that $X = \bigcup_{g\in G} gU$. Then G is generated by $S := \{g \in G \mid gU \cap U \neq \emptyset\}$.*

Proof. Let $H := \langle S \rangle$. Then the two sets HU and $(G-H)U$ are both open. In addition, they are disjoint: Suppose we had $hU \cap fU \neq \emptyset$ for some $h \in H$ and $f \in G-H$. Then $f^{-1}h \in H$ and therefore $f \in H$ contrary to our assumption.

As X is connected, the set $(G-H)U$ is empty as the other one cannot be empty. Hence $G - H = \emptyset$. **q.e.d.**

There are variations of this lemma, e.g., on group presentations. We will encounter them later.

Corollary 1.51. *A finite index subgroup of a finitely generated group is finitely generated.*

Proof. Let G be a group with finite generating set Σ and let H be a subgroup of finite index. Let $r_1, \ldots, r_r$ be a list containing a representative for each H-coset in G. Hence every vertex of the Cayley graph $\Gamma_\Sigma(G)$ lies in the orbit of one of the r_i. Let U be the union of open stars of the r_i. The hypotheses of (1.50) are obviously satisfied. Hence

$$\{h \in H \mid hU \cap U \neq \emptyset\} = \{h \in H \mid hr_i = gr_j \text{ for some } g \in \Sigma \cup \{1\}\}$$

generates H. But that set is finite. **q.e.d.**

Exercise 1.52. Let $\Gamma = \Gamma_\Sigma(G)$ be a Cayley graph for the infinite, finitely generated group G with respect to the finite generating system Σ. Show that Γ contains a bi-infinite short-lex geodesic (defined below).

Any edge path in Γ reads a word in $\Sigma \uplus \Sigma^{-1}$: while you are moving along the path, you pick up the labels of the edges you are going along, when you move with the direction of the edge you read the label, when you are going against the directed edge in Γ you read its inverse.

Fix an order on the set Σ. This induces an ordering on the set of word with letters from Σ: shorter words precede longer words and you use the lexicographic order to break ties. Regarding inverses as lower case variants of the capital letters in Σ, we actually have an order on words in $\Sigma \uplus \Sigma^{-1}$. Every group element is represented by a unique short-lex minimal word. Hence any to vertices in Γ are joined by a unique short-lex minimal edge path. We call those paths short-lex geodesic segments. Note that they are, in fact, geodesic segments.

Now a (bi-infinite) short-lex geodesic is a (bi-infinite) edge path such that every finite sub path is a short-lex geodesic.

Hint: First prove that Γ contains a bi-infinite geodesic.

1.4.1 Ends

Definition 1.53. A diagram (of sets and maps) is a directed graph D whose vertices v are labeled by sets M_v and whose edges $\overrightarrow{e}$ are labeled by maps $f_{\overrightarrow{e}} : M_{\iota(\overrightarrow{e})} \to M_{\tau(\overrightarrow{e})}$. The inverse limit of D is the set

$$\varprojlim D := \left\{ (m_v \in M_v)_{v \in \mathcal{V}(D)} \,\middle|\, f_{\overrightarrow{e}}\left(m_{\iota(\overrightarrow{e})}\right) = m_{\tau(\overrightarrow{e})} \text{ for all } \overrightarrow{e} \in \mathcal{E}(D) \right\}$$

Note that there are natural maps $\varprojlim D \to M_v$ for all vertices $v \in D$ and all triangles

$$\begin{array}{ccc} \varprojlim D & \rightarrow & M_{\tau(\overrightarrow{e})} \\ \downarrow & \nearrow f_{\overrightarrow{e}} & \\ M_{\iota(\overrightarrow{e})} & & \end{array}$$

commute.

Definition 1.54. Let X be a topological space. For any two nested compact subsets, $C \subseteq D \subseteq X$, we have a natural map

$$\pi_0(X - D) \to \pi_0(X - C).$$

As compact subsets in X form a directed set, we can write the inverse limit

$$\partial_\infty X := \varprojlim_{C \subseteq X} \pi_0(X - C)$$

The elements of the set $\partial_\infty X$ are called the ends of X.

Example 1.55. The two Cayley graphs of C_∞ both have precisely two ends.

Observation 1.56. *This construction is functorial, so homeomorphisms of X induce bijections of $\partial_\infty X$ and we have a group homomorphism*

$$\operatorname{Homeo}(X) \to \operatorname{Perm}(\partial_\infty X).$$

In particular, if $X = \Gamma_\Sigma(G)$ is a (left) Cayley graph for a group G, there is a natural action of G on $\partial_\infty \Gamma_\Sigma$ turning the set of ends into a G-set.

Exercise 1.57. The number of ends in a Cayley graph is 0, 1, 2, or ∞: Let $\Gamma := \Gamma_\Sigma(G)$ be the Cayley graph for the group G with with respect to the finite generating set Σ. Show that if Γ has finitely many ends, then the number of ends is ≤ 2. Hint: Assume Γ has three ends. Then there should be a central region where theses ends get tied up. But a Cayley graph looks homogeneous as there is a vertex transitive group action, hence there cannot be a distinguished region.

Exercise 1.58. Given the same setup as in (1.57), show that the number of ends (0, 1, 2, or ∞) is independent of the choice of the finite generating system Σ.

The following theorem relates the geometry of a Cayley graph to a purely algebraic property of a group. In this respect it is like Gromov's theorem 1.70. But it is way simpler, and it is about C_∞.

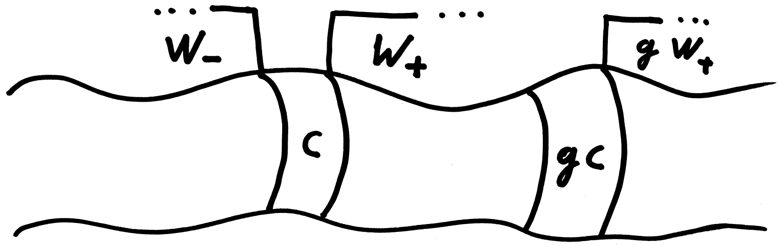
Theorem 1.59. *A group has two ends if and only if it is virtually* C_∞.

Proof. That a group which is virtually C_∞ has two ends is easy. We only proof the converse. So let Σ be a generating set for G such that the corresponding Cayley graph Γ has two ends.

Our group G acts on $\partial_\infty\Gamma$ and the kernel of this action has index ≤ 2 in G whence we may assume without loss of generality that G fixes both ends of Γ.

Since Γ has two ends, there is a compact subset C such that $\Gamma - C$ has exactly two infinite components W_- and W_+. We add all finite component of $\Gamma - C$ and can henceforth assume that the two infinite components are all there is in $\Gamma - C$. Since our space is infinite, there is an element $g \in G$ that moves C off itself: $gC \cap C = \emptyset$.

First, we show that g has infinite order. The translate gC lies in one of the components. We assume $gC \subset W_+$. Then $gW_+ \subsetneq W_+$ for otherwise g would swap the ends of Γ.



Hence we find

$$W_+ \supsetneq gW_+ \supsetneq g^2 W_+ \supsetneq \cdots$$

and it follows that g has infinite order.

Let D be a compact subset containing C and its translate gC such that $\Gamma - D$ has exactly two components both of which are infinite. It follows that

$$\Gamma = W_- \cup D \cup gW_+.$$

We infer, that $\Gamma = \bigcup_{i \geq 0} g^i W_-$ whence $\bigcap_{i \geq 0} g^i W_+ = \emptyset$. Similarly, $\bigcap_{i \leq 0} g^i W_- = \emptyset$. Moreover, for any $i > 0$,

$$\Gamma = g^{-i} W_- \cup \bigcup_{-i \leq j < i} g^j D \cup g^i W_+$$

whence

$$\Gamma = \bigcup_{s \in \mathbb{Z}} g^s D.$$

This, however, implies that D contains a representative for each coset in $\langle g \rangle \backslash G$. Hence $\langle g \rangle$ has finite index in G. **q.e.d.**

1.4.2 Growth

Definition 1.60. The growth function β_Σ of G relative to Σ is defined by

$$\beta_\Sigma(n) := \mathrm{vol}\left(\mathbb{B}_n(1_G)\right)$$

where $\mathbb{B}_n(1_G)$ is the ball of radius n in the Cayley graph centered at $1_G \in G$ and volume is measured by counting vertices.

Example 1.61. For the two Cayley graphs of C_∞, we find:

$$\beta_{\{1\}}(n) = 2n+1$$

and

$$\beta_{\{2,3\}}(n) = \begin{cases} 1 & n=0 \\ 5 & n=1 \\ 6n+1 & n \geq 2 \end{cases}$$

Note that any generating set for C_∞ will yield an ultimately linear growth function.

Definition 1.62. Let β and β' be two functions defined on $\mathbb{N}$. We say β' weakly dominates β if there are constants L and K such that

$$\beta(n) \leq L\beta'(Ln+K)+K.$$

We write $\beta \preceq \beta'$. We say that β and β' are weakly equivalent if they weakly dominate one another:

$$\beta \sim \beta' :\Longleftrightarrow \beta \preceq \beta' \ \& \ \beta' \preceq \beta$$

Remark 1.63. Weak domination is transitive, and weak equivalence is an equivalence relation.

Observation 1.64. *Let Σ and Ξ be two finite generating sets for G, then the growth functions β_Σ and β_Ξ are weakly equivalent. To see this write the elements of Ξ as words in Σ let l be the maximum length that occurs in this list of words. Then any n-ball in Γ_Ξ is contained in the ln-ball in Γ_Σ which proves $\beta_\Xi \preceq \beta_\Sigma$.* **q.e.d.**

Exercise 1.65. Let G be finitely generated and H be a subgroup of finite index, which is, therefore, finitely generated, as well. Show that the growth functions for these two groups are weakly equivalent.

Observation 1.66. *For an infinite group with generating set Σ, we have*

$$n \leq \beta_\Sigma(n) \leq \sum_{i=0}^{n} (2|\Sigma|)^i$$

Since there are at most as many elements in the n-ball as there are words of length $\leq n$ in the generators (and their inverses!). The lower bound follows from the fact that the Cayley graph is connected and infinite. **q.e.d.**

So growth in groups is somewhere between linear and exponential. One distinguishes three cases:

Definition 1.67. A finitely generated group is of polynomial growth if its growth function is weakly dominated by a polynomial. It is of exponential growth if it weakly dominates an exponential function. Otherwise it is of intermediate growth.

Exercise 1.68. Show that a finitely generated group with infinitely many ends has exponential growth.

Proposition 1.69. *Groups of subexponential growth are amenable.*

Proof. We will show that a group G of sublinear growth has a Følner sequence consisting of balls $\mathbb{B}_n(1_G)$. For assume this was not the case, then there is an $\varepsilon > 0$ such that for any ball $\mathbb{B}_n$ there is an element $x \in \Sigma$ such that

$$\frac{|x\mathbb{B}_n \triangle \mathbb{B}_n|}{|\mathbb{B}_n|} > \varepsilon.$$

Since $\mathbb{B}_{n+1}$ contains as well $\mathbb{B}_n$ as $x\mathbb{B}_n$, we have:

$$\beta_\Sigma(n+1) = |\mathbb{B}_{n+1}| \geq |x\mathbb{B}_n \cup \mathbb{B}_n| \geq \left(1 + \frac{\varepsilon}{2}\right)|\mathbb{B}_n| = \left(1 + \frac{\varepsilon}{2}\right)\beta_\Sigma(n)$$

From this, it is obvious that G has exponential growth. **q.e.d.**

In particular, groups of polynomial growth are amenable. However, a very deep theorem says that we already knew that:

Theorem 1.70 (Gromov [Grom81]). *A finitely generated group has polynomial growth if and only if it is virtually nilpotent.*

This is way too deep for this exposition, and fortunately it is not a statement about the infinite cyclic group. However, there is a characterization of C_∞ by means of its growth:

Theorem 1.71. *A group has linear growth if and only if it is virtually C_∞.*

Before we prove this, we need a geometric lemma on growth rates.

Lemma 1.72. *Let H be a subgroup of G of infinite index. Let Σ be a finite generating system for H and Ξ a finite generating set for G that contains Σ. Then:*

$$\beta_\Xi(2n) \geq (n+1)\,\beta_\Sigma(n) \qquad \textit{for all } n \in \mathbb{N}$$

In particular, if H has polynomial growth of degree d, then G has at least polynomial growth of degree $d+1$.

Proof. Let $X := H\backslash\Gamma_\Xi(G)$ be the space of orbits of vertices in $\Gamma_\Xi(G)$ under the action of H. We turn this into a metric space by defining the distance of two orbits to be the minimum distance of two representatives. Note that we can choose one of them at our will. A pair of representatives realizing the distance shows that two points of distance n in X are joined by a path of length n. It follows that the ball $\mathbb{B}_n(H) \subset X$ contains at least $n+1$ points $x_0, \ldots, x_n$. Each of these vertices has a representative $g_i \in \mathbb{B}_n(1_G) \subseteq \Gamma_\Xi$. Now we consider translates of the balls in $\Gamma_\Sigma(H)$. We find

$$\mathbb{B}_{2n}(1_G) \supseteq \bigcup_{i=0}^{n} \mathbb{B}_n(1_H)\,g_i$$

and the inequality follows from the fact, that the union is disjoint. **q.e.d.**

Proof of Theorem 1.71. Since finite groups have "constant growth" which is not linear, we have to consider infinite groups only. One direction is obvious: If a group has an infinite cyclic subgroup of finite index, it has linear growth. So we have to show the converse: Any infinite group G of linear growth contains an infinite cyclic subgroup of finite index. By the preceding lemma, it suffices to show that there is an element of infinite order in G since the infinite cyclic subgroup it generates cannot have infinite index in G.

Let us fix a finite generating set Σ for G. We know by (1.52) that there is a bi-infinite short-lex geodesic inside the Cayley graph $\Gamma := \Gamma_\Sigma(G)$.

First we prove that this geodesic is ultimately periodic at its "right end": There is a constant L such that, for infinitely many n, we have:

$$\mathrm{vol}(\mathbb{B}_n) - \mathrm{vol}(\mathbb{B}_{n-1}) \leq L \tag{1}$$

Consider a finite subset of vertices $W = \{v_1, v_2, \ldots, v_r\}$ on the geodesic with $r > L$. For any n that satisfies (1), there is a pair of distinct vertices $v, w \in W$ such that the geodesic segments of length n starting at these vertices and extending to the right both read the same word. Since there are infinitely many such n, there is a pair of vertices for which this happens infinitely many times. It follows that the geodesic is ultimately right periodic.

The group element represented by the period obviously has infinite order. So we have our desired cyclic subgroup of finite index. **q.e.d.**