

## 2 Free Groups of Finite Rank

**Definition 2.1.** Let  $M$  be a set. The free group  $F_M$  generated by  $M$  is a group that contains  $M$  as a subset and that is uniquely determined up to unique isomorphism by the universal property that any map  $f : M \rightarrow G$  from  $M$  to any group  $G$  extends to a unique group homomorphism  $\varphi_f : F_M \rightarrow G$ .

**Remark 2.2.** The elements of  $F_M$  are reduced words in the alphabet  $M \uplus M^{-1}$ . Multiplication is concatenation of words followed by reduction, i.e., cancellation of subwords  $mm^{-1}$  until no longer possible. The empty word serves as the trivial element. Of course there are some claims here to be proved, but you are supposed to have done this already in some other class.

### 2.1 Free Constructions

**Definition 2.3.** Let  $D$  be a diagram of groups  $G_v$  and homomorphisms  $\varphi_{\vec{e}} : G_{\iota(\vec{e})} \rightarrow G_{\tau(\vec{e})}$ . The direct limit of  $D$  is a group  $\varinjlim D$  together with homomorphisms  $\iota_v : G_v \rightarrow \varinjlim D$  such that

1. all triangles

$$\begin{array}{ccc} \varinjlim D & \leftarrow & G_{\tau(\vec{e})} \\ \uparrow & \nearrow \varphi_{\vec{e}} & \\ G_{\iota(\vec{e})} & & \end{array}$$

commute.

2. Given any other group  $H$  together with a family of homomorphism  $\varphi_v : G_v \rightarrow H$  making the corresponding triangles (as in 1) commutative, there is a unique homomorphism  $\pi : \varinjlim D \rightarrow H$  such that all triangles

$$\begin{array}{ccc} G_v & \xrightarrow{\varphi_v} & H \\ \downarrow \iota_v & \nearrow \pi & \\ \varinjlim D & & \end{array}$$

commute.

**Exercise 2.4.** The usual category theoretic nonsense proves uniqueness of direct limits for free. Show that direct limits exist in the category of groups and homomorphisms.

**Definition 2.5.** Let  $C \hookrightarrow A$  and  $C \hookrightarrow B$  be two monomorphisms. The amalgamated product  $A *_C B$  is the direct limit of the diagram

$$A \leftarrow C \rightarrow B$$

The free product  $G * H$  of two groups is their amalgamated product along the trivial group:

$$G * H := \varinjlim (G \leftarrow 1 \rightarrow H)$$

These cases arise naturally in topology.

**Example 2.6 (van Kampen).** Let  $X$  be a path connected topological space with base point. Assume we are given a open cover  $X = U \cup V$  such that  $U$ ,  $V$ , and  $X \cap V$  are path connected subsets of  $X$  that contain the base point. Then

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

**Example 2.7.** The free group on  $n$  generators is the free product of  $n$  copies of  $C_\infty$ :

$$F_n := \bigstar_{i=1}^n C_\infty$$

In general, the free group generated the set  $M$  is

$$F_M := \bigstar_{m \in M} C_\infty$$

**Observation 2.8.** *As a consequence of van Kampen's theorem, we see that a group is free if and only if it is the fundamental group of a graph.*

**Corollary 2.9.** *Subgroups of free groups are free.*

**Proof.** Let  $F$  be a free group. Then  $F$  is the fundamental group of a graph. Its subgroups occur as fundamental groups of covers. However, any cover of a graph is a graph. Hence any subgroup of  $F$  is the fundamental group of a graph and, therefore, free. **q.e.d.**

**Observation 2.10.** Obviously, we can construct very large covers. In particular,  $F_2$  contains a copy of  $F_{\mathbb{N}}$ .

**Observation 2.11.** From their geometric realization, we can read off a presentation for free groups:

$$F_n = \langle x_1, \dots, x_n \rangle$$

Each generator corresponds to a loop in a wedge of  $n$  circles. The Cayley graph of  $F_n$  corresponding to this system of free generators is the universal cover of the wedge of circles. It is a tree.

**Corollary 2.12.** Non-abelian free groups have exponential growth. **q.e.d.**

**Exercise 2.13 (Schreier's Index Formula).** Let  $G$  be a subgroup of  $F_n$  of finite index  $s$ . Prove that  $G$  is isomorphic to  $F_{s(n-1)-1}$ .

## 2.2 How to Detect Free Groups?

**Lemma 2.14 (Ping Pong Lemma).** Let  $G$  be a group acting on a set  $M$ . Suppose  $H_1$  and  $H_2$  are two subgroups of  $G$  with cardinalities at least 3 and 2, respectively. Let  $H$  be the subgroup generated by  $H_1$  and  $H_2$ .

Assume that there are two non-empty subsets  $S_1$  and  $S_2$  in  $M$  such that

$$S_2 \not\subseteq S_1$$

$$gS_2 \subseteq S_1 \quad \text{for all } g \in H_1, g \neq 1$$

$$gS_1 \subseteq S_2 \quad \text{for all } g \in H_2, g \neq 1$$

Then  $H$  is isomorphic to the free product  $H_1 * H_2$ .

**Proof.** We have to show that a product whose factors are all non-trivial and alternately taken from the groups  $H_1$  and  $H_2$  is non-trivial. We start by considering a product of odd length

$$w = a_1 b_1 a_2 b_2 \cdots b_{r-1} a_r$$

wherein  $a_i \in H_1 - \{1\}$  and  $b_i \in H_2 - \{1\}$ . We have

$$wS_2 = a_1 b_1 a_2 b_2 \cdots b_{r-1} a_r S_2 \subseteq a_1 b_1 a_2 b_2 \cdots b_{r-1} S_1 \cdots \subseteq a_1 S_2 \subseteq S_1$$

whence  $w$  acts non-trivially as  $S_2 \not\subseteq S_1$ .

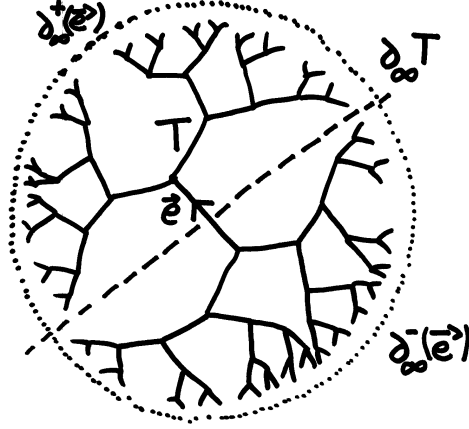
For a word that starts and ends with a letter from  $H_2$ , we conjugate it by a non-trivial element from  $H_1$ . As conjugation preserves being trivial or non-trivial, we are reduced to the first case.

For a word of even length, only one boundary letter is in  $H_1$ . Let this letter be  $a \in H_1$ . Conjugation by an element of  $H - \{1, a\}$  reduces us to the first case. Here, we need that  $H_1$  has at least three elements. **q.e.d.**

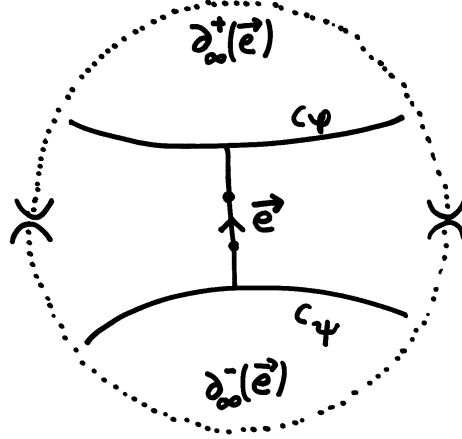
**Example 2.15.** Let  $T$  be a tree. An automorphism of  $T$  is called hyperbolic if it stabilizes a bi-infinite geodesic in  $T$  upon which it acts as a non-trivial shift. Then, this geodesic  $C_\varphi$  is unique and called the axis of the automorphism.

Let  $\varphi$  and  $\psi$  be two hyperbolic automorphisms of  $T$  with disjoint axes. Then  $\langle \varphi, \psi \rangle$  is free.

**Proof.** We will study the action on the set of ends  $\partial_\infty T$ . Note that each oriented edge  $\vec{e}$  defines a decomposition  $\partial_\infty T = \partial_\infty^+ \vec{e} \sqcup \partial_\infty^- \vec{e}$  as in the picture:



Let  $\vec{e}$  be an edge on the geodesic joining  $C_\varphi$  and  $C_\psi$ .



Then any non-trivial power of  $\varphi$  will suck all of  $\partial_\infty^-(\vec{e})$  into  $\partial_\infty^+(\vec{e})$  and any non-trivial power of  $\psi$  will take  $\partial_\infty^+(\vec{e})$  into  $\partial_\infty^-(\vec{e})$ . Hence

$$\langle \varphi, \psi \rangle = \langle \varphi \rangle * \langle \psi \rangle$$

which is a non-abelian free group.

**q.e.d.**

**Exercise 2.16.** Suppose  $\varphi$  and  $\psi$  are two hyperbolic automorphisms of a tree  $T$  whose axes have a finite intersection. Show that sufficiently high powers  $\varphi^k$  and  $\psi^l$  generate a free group.

**Exercise 2.17.** Show that  $F_2$  embeds into  $C_2 * C_3$ . Here,  $C_n$  is the cyclic group of order  $n$ .

**Example 2.18.** The two matrices  $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$  generate a free group inside  $\text{SL}_2(\mathbb{Z})$ .

**Proof.** Each of the two matrices generates an infinite cyclic subgroup inside  $\text{SL}_2(\mathbb{Z})$ . So we have to consider the group generated by the subgroups

$$H_1 := \left\{ \begin{pmatrix} 1 & 2s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{Z} \right\}$$

and

$$H_2 := \left\{ \begin{pmatrix} 1 & 0 \\ 2s & 1 \end{pmatrix} \mid s \in \mathbb{Z} \right\}$$

Put

$$S_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid |x| > |y| \right\}$$

and

$$S_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid |x| < |y| \right\}$$

To verify the hypotheses of the Ping Pong Lemma, we compute

$$\begin{pmatrix} 1 & 2s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2sy \\ y \end{pmatrix}$$

whence we have to show that for  $|x| < |y|$  it follows that  $|x + 2sy| > |y|$ . This is obvious from the triangle inequality:

$$|x + 2sy| > |2sy| - |x| = |y| + ((2s - 1)|y| - |x|)$$

The other hypothesis is checked analogously.

**q.e.d.**

**Corollary 2.19.** *Finitely generated free groups are linear and residually finite.*

**Proof.** As we have seen,  $F_2$  embeds into  $SL_2(\mathbb{Z})$ , which is residually finite as every non-trivial element survives in a factor  $SL_2(\mathbb{Z}/p\mathbb{Z})$  where  $p$  is a sufficiently large prime number. **q.e.d.**

A way more sophisticated application of the Ping Pong Lemma (or better: a slight variation of it) are the following celebrated theorems due to J. Tits:

**Theorem 2.20 (Tits [Tits72]).** *Over a field of characteristic 0, a linear group either is virtually solvable (i.e., it is small) or has a non-abelian free subgroup (i.e., it is big).*

**Theorem 2.21 (Tits [Tits72]).** *A finitely generated linear group either is virtually solvable or has a non-abelian free group.*

These results (and many others that followed) motivates

**Definition 2.22.** A group  $G$  satisfies the Tits Alternative if each finitely generated subgroups either is virtually solvable or contains a non-abelian free group.

**Remark 2.23.** Tits' result states that linear groups satisfy the Tits-Alternative. Another example would be  $\text{Out}(F_n)$ .

**Exercise 2.24.** Show that a virtually solvable group cannot contain a non-abelian free group.

**Exercise 2.25.** By Tits' theorem, the group  $SO_3(\mathbb{R})$  has a non-abelian free subgroup. Find an embedding of  $F_2 \hookrightarrow SO_3(\mathbb{R})$ .

## 2.3 Kazhdan's Property (T) and Amenability

We already observed (1.45) that non-abelian free groups do not have Kazhdan's property (T). However, although property (T) and amenability are mutually exclusive for infinite groups, they are not complementary.

**Theorem 2.26.** *Non-abelian free groups are not amenable.*

**Proof.** We only do the argument for  $F_2 = \langle x, y \rangle$ . Consider the following bounded functions on  $F_2$ :

$$\begin{aligned} f_1 &: w \mapsto \begin{cases} 1 & \text{if } w = 1 \\ 0 & \text{otherwise} \end{cases} \\ f_x &: w \mapsto \begin{cases} 1 & \text{if } w \text{ starts with } x \\ 0 & \text{otherwise} \end{cases} \\ f_{\bar{x}} &: w \mapsto \begin{cases} 1 & \text{if } w \text{ starts with } \bar{x} \\ 0 & \text{otherwise} \end{cases} \\ f_y &: w \mapsto \begin{cases} 1 & \text{if } w \text{ starts with } y \\ 0 & \text{otherwise} \end{cases} \\ f_{\bar{y}} &: w \mapsto \begin{cases} 1 & \text{if } w \text{ starts with } \bar{y} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Obviously,

$$1 = f_1 + f_x + f_{\bar{x}} + f_y + f_{\bar{y}}.$$

Now we form

$$h := f_1 + \bar{x}f_x + xf_{\bar{x}} + \bar{y}f_y + yf_{\bar{y}}$$

We have

$$h(w) = \begin{cases} 5 & \text{if } w = 1 \\ 3 & \text{otherwise} \end{cases}$$

This clearly rules out the possibility of an invariant measure on this group. **q.e.d.**

The trick in this proof motivates



**Definition 2.27.** A paradoxical partition of unity is a finite partition of unity

$$1 = f_1 + \cdots + f_r$$

together with an  $r$ -tuple of group elements  $g_1, \dots, g_r$  such that:

- Each  $f$  is a bounded function.
- There is an  $\varepsilon > 0$  satisfying

$$g_1 f_1 + \cdots + g_r f_r \geq 1 + \varepsilon$$

**Observation 2.28.** *No amenable group admits a paradoxical partition of unity.*

### 2.3.1 Equivalent Formulations for Amenability

**Definition 2.29.** Let  $\Gamma$  be a (multi)-graph. A flow on  $\Gamma$  is a map  $\Phi : \vec{\mathcal{E}}(\Gamma) \rightarrow \mathbb{R}$  on the oriented edges satisfying

$$\Phi(\vec{e}^{\text{op}}) = -\Phi(\vec{e})$$

For any vertex  $v$  of  $\Gamma$ , the net production is

$$P_v := \sum_{v=\iota(\vec{e})} \Phi(\vec{e})$$

A vertex is called a source if its net production is  $> 0$ , a sink if its net production is  $< 0$  and it is called balanced if its net production is 0. A flow  $\Phi$  is called  $\varepsilon$ -productive if every vertex has net production  $\geq \varepsilon$ .

A capacity on  $\Gamma$  is a map  $C : \vec{\mathcal{E}}(\Gamma) \rightarrow \mathbb{R}^+$ . The flow  $\Phi$  is bounded by the capacity  $C$  if  $\Phi(\vec{e}) \leq C_{\vec{e}}$  for each oriented edge  $\vec{e}$ . Note that a geometric edge can have different capacities in its two directions.

Let  $v$  and  $w$  be two vertices in  $\Gamma$ . A cut is a set of oriented edges in  $\Gamma$  such that any path from  $v$  to  $w$  has to pass

through at least one edge in the cut thereby respecting the given orientation of that edge. (It may pass through other edges in the cut in any representation, but at least one edge has to be crossed “in the right direction”.)

For any set of vertices  $W \subseteq \Gamma$ , its boundary  $\partial W$  is the set of edges with one endpoint inside  $W$  and the other endpoint outside  $W$ .

For finite graphs, we have the following nice correspondence:

**Theorem 2.30 (max flow equals min cut).** *In a finite graph with one source  $O$  and one sink  $I$ , the maximum net productivity of any flow bounded by a capacity  $C$  is the minimum total capacity of edges in a cut.*

**Proof.** First observe that for any any flow  $\Phi$  and any minimal cut set  $E_{\min}$ ,

$$P_{\Phi}(O) = -P_I = \sum_{\vec{e} \in E_{\min}} \Phi(\vec{e}) \leq \sum_{\vec{e} \in E_{\min}} C_{\vec{e}}$$

holds. So we only have to prove that there is a flow whose throughput realizes the capacity of a minimal cut.

The set of all flows on  $\Gamma$  bounded by  $C$  with unique source at  $O$  and unique sink at  $I$  is compact and the net production of the source is a continuous function. So let  $\Phi$  be a maximum flow. Consider the set of vertices

$$W := \left\{ v \in \Gamma \left| \begin{array}{l} \text{there is a path} \\ O = v_0 \xrightarrow{\vec{e}_1} v_1 \xrightarrow{\vec{e}_2} \dots \xrightarrow{\vec{e}_r} v_r = v \\ \text{such that } \Phi(\vec{e}_i) < C_{\vec{e}_i} \end{array} \right. \right\}$$

reachable from the source by means of a non-saturated path.

Clearly,  $I \notin W$ , since otherwise we could increase the flow along a non-saturated path from  $O$  to  $I$ .

Hence

$$E := \left\{ \vec{e} \in \vec{\mathcal{E}}(\Gamma) \mid \iota(\vec{e}) \in W \text{ and } \tau(\vec{e}) \notin W \right\}$$

is a cut. In addition, every edge in  $E$  is saturated, for otherwise its end point would be in  $W$ . The capacity of  $E$  clearly equals the net production of the source. **q.e.d.**

**Theorem and Definition 2.31.** *Let  $\Gamma$  be a graph and  $\varepsilon \geq 0$ . Then the following are equivalent:*

1. *For any finite set of vertices  $W \subseteq \Gamma$ ,*

$$\frac{|\partial W|}{|W|} \geq \varepsilon$$

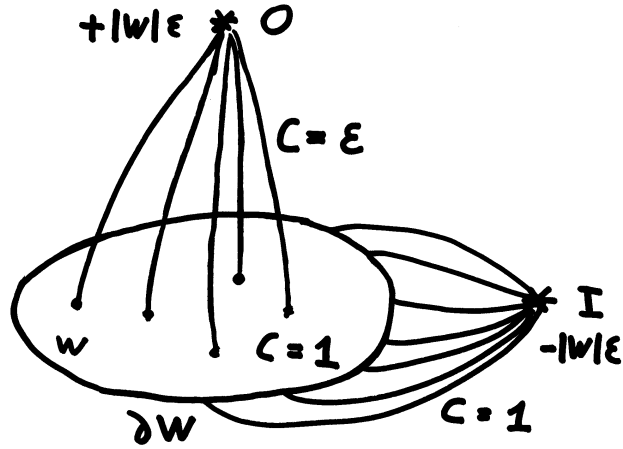
2. *There is an  $\varepsilon$ -productive flow  $\Phi$  bounded by capacity 1.*

*If  $\Gamma$  satisfies the condition 1 above for an  $\varepsilon > 0$ , we say  $\Gamma$  satisfies a strong isoperimetric inequality.*

**Proof.** Let  $W$  be a finite set of vertices in  $\Gamma$ . It is obvious that an  $\varepsilon$ -productive flow will have to move a total mass of  $\varepsilon|W|$  out off this area. However, as all edges have capacity bounded by one, there must be at least that many edges leaving  $W$ . Hence (2) clearly implies (1).

To prove the other direction, we employ the following strategy: First we use the min-flow-max-cut theorem to prove that for any finite set of vertices  $W$ , we can find flow  $\Phi_W$  that is bounded by 1 yet “looks”  $\varepsilon$ -productive on  $W$ . In a second step, we use an ultrafilter construction, to patch these flows together.

Let  $W$  be a finite set of vertices in  $\Gamma$ . We collapse the complement of  $W$  to a single vertex  $I$  (the sInk). We assign a capacity of 1 to all edges. Now, we introduce a new vertex  $O$  (the sOurce) which we connect by a new edge to any vertex in  $W$ ; to all the new edges we assign the capacity  $\varepsilon$ .



Let us compute the minimal capacity of a cut separating the source  $O$  from the sink  $I$ . Any cut splits  $W$  into two parts: the set  $W_O$  of vertices still connected to the source  $O$  after removing the cut, and the set  $W_I$  of vertices still connected to the sink afterward. Obviously, all those  $\varepsilon$ -capacity edges connecting  $O$  to points in  $W_I$  are in the cut set. They contribute a total capacity of  $\varepsilon|W_I|$ . On the other hand, all of the boundary  $\partial W_O$  belong to the cut, as well. Each of these edges has a capacity of 1, but there are at least  $\varepsilon|W_O|$  of these because of the isoperimetric inequality (1) as applied to  $W_O$  – recall that we want to prove (2) from (1).

So the total capacity of any cut set is at least  $\varepsilon|W_I| + \varepsilon|W_O| = \varepsilon|W|$  whence there is a flow  $\Phi_W$  of this throughput by the max-flow-min-cut theorem.

We delete the source and the  $\varepsilon$ -capacity edges, undo the collapse which created the sink, and re-interpret  $\Phi_W$  as a flow on  $\Gamma$  where all edges outside are assigned 0. This flow  $\Phi_W$  is bounded by 1 and has net production  $P_v = \varepsilon$  for every vertex in  $W$ . This completes the first step of the construction.

The second step is to construct a flow that has net production  $\varepsilon$  everywhere. To do this, pick your favorite ultrafilter  $\mathcal{U}$  refining the coinitial filter on the directed set  $D$  of finite vertex sets in  $\Gamma$ . For each element  $W$  in this index set  $D$ , already have a flow  $\Phi_W$ . So if we want to define the global flow  $\Phi$  on an

oriented edge  $\vec{e}$ , we put

$$\Phi(\vec{e}) := \mathcal{U}\text{-}\lim \Phi_W(\vec{e})$$

It is a routine matter to check that this does the job. **q.e.d.**

**Definition 2.32.** A generating set  $\Sigma$  for a group  $G$  is symmetric if  $\Sigma = \Sigma^{-1}$ . For symmetric generating sets, one usually uses the reduced (right) Cayley graph wherein the edges  $g \xrightarrow{x} xg$  and  $xg \xrightarrow{x^{-1}} g$  are considered as opposite orientations of one underlying geometric edge.

**Corollary 2.33.** *Let  $G$  be a finitely generated group with finite symmetric generating set  $\Sigma \not\ni 1$ . Then the following are equivalent:*

1.  $G$  is amenable.
2.  $G$  has a Følner sequence.
3. The reduced Cayley graph  $\Gamma$  does not satisfy a strong isoperimetric inequality.
4. There is no productive bounded flow on the reduced (right) Cayley graph.
5.  $G$  does not have a paradoxical partition of unity.

In fact, one could use ordinary (unreduced, left) Cayley graphs instead. We confine ourselves to reduced right Cayley graphs only to avoid technical issues.

**Proof.** We already proved  $(2) \implies (1)$  and  $(1) \implies (5)$ . The implication  $(3) \implies (2)$  is immediate, and  $(4) \implies (3)$  follows from (2.31).

The remaining implication  $(5) \implies (4)$  is done by re-interpreting a flow as a paradoxical partition: Let  $\Phi$  be an  $\varepsilon$ -productive flow bounded by 1 on  $\Gamma$ . Define

$$f_\sigma(g) := \begin{cases} -\Phi\left(g \xrightarrow{x} xg\right) & \text{if } \Phi\left(g \xrightarrow{x} xg\right) < 0 \\ 0 & \text{otherwise} \end{cases}$$

be the outbound flow from  $g$  to  $\sigma g$ . Moreover, put

$$f_1(g) := |\Sigma| - \sum_{\sigma \in \Sigma} f_\sigma(g).$$

Obviously, we have a partition of the constant function  $|\Sigma|$ .

Now compute

$$\begin{aligned} f_1(g) + \sum_{\sigma \in \Sigma} \sigma^{-1} f_{\sigma^{-1}}(g) &= |\Sigma| + \sum_{\sigma \in \Sigma} f_{\sigma^{-1}}(\sigma g) - f_\sigma(g) \\ &= |\Sigma| + \sum_{\sigma \in \Sigma} \Phi\left(g \xrightarrow{\sigma} \Sigma g\right) \\ &\geq |\Sigma| + \varepsilon \end{aligned}$$

Division by  $|\Sigma|$  yields a paradoxical partition of unity. **q.e.d.**

**Remark 2.34.** The usual way of proving these equivalences establishes separately

$$(1) \iff (5) \iff (4)$$

and

$$(1) \iff (2) \iff (3)$$

This way, however, no relation in the numerical constants of the isoperimetric inequality and the productivity of the flow is established. Moreover, the implication  $(1) \implies (2)$  involves functional analysis.

**Exercise 2.35.** A paradoxical decomposition of a group  $G$  is a partition

$$G = S_1 \uplus \cdots \uplus S_r \uplus T_1 \uplus \cdots \uplus T_s$$

such that there are group elements  $g_1, \dots, g_r$  and  $h_1, \dots, h_s$  such that

$$G = g_1 S_1 \uplus \cdots \uplus g_r S_r$$

and

$$G = h_1 T_1 \uplus \cdots \uplus h_s T_s$$

Prove that  $F_2$  has a paradoxical decomposition.

**Remark 2.36.** As a matter of fact, a group has a paradoxical decomposition if and only if it is not amenable. As a criterion to check this, however, using flows or a paradoxical partition of unity is more easy.

## 2.4 The Hanna Neumann Conjecture

**Definition 2.37.** A group  $G$  has the finite intersection property if the intersection of any two finitely generated subgroups in  $G$  is finitely generated.

**Theorem 2.38 (Howson [How54]).** *Free groups enjoy the finite intersection property.*

By now, there are many proofs of this theorem. The given here is stolen from [Shor90].

**Definition 2.39.** Let  $G$  be a group with finite generating set  $\Sigma$ . A subgroup  $H \leq G$  is quasi-convex with respect to  $\Sigma$  if there is a constant  $R \geq 0$  such that every geodesic path in the Cayley graph  $\Gamma_\Sigma(G)$  joining two points in  $H$  lies in an  $R$ -neighbourhood of  $H$ . That is, every point on such a path has distance  $\leq R$  to at least one point in  $H \subset \Gamma$ .

**Example 2.40.** Any finitely generated subgroup  $H$  of a free group is quasi-convex with respect to the standard generators: Let  $B$  be a ball in the Cayley tree  $\Gamma$  centered at 1 containing all generators of  $H$ . The union  $HB$  is connected and hence a subtree. Any geodesic joining two point of  $H$  in  $\Gamma$  actually lies in  $HB$ . The constant  $R$ , therefore can be chosen to be the radius of  $B$ .

**Proposition 2.41.** *Quasi-convex subgroups are finitely generated.*

**Proof.** Let  $G$ ,  $H$ ,  $\Sigma$ , and  $R$  be as in the definition (2.39), and let  $B$  be the open ball in  $\Gamma = \Gamma_\Sigma(G)$  of radius  $R+1$ . It is easy to see that

$$X = HB \subseteq \Gamma$$