MIDTERM EXAM, MATH 3210, FALL 2010

Problem 0.1. *A function* $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ *is said to be homogeneous of degree* p*, if* $f(\lambda x) = \lambda^p f(x)$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$ *and* $\lambda > 0$.

(i) Show that if f *is homogeneous of degree* p *and smooth, then*

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = pf(x).$$

(ii) Let then $\alpha = \sum_{i=1}^{n} f_i dx_i$ *be a closed* 1*-form, whose coefficients* f_i *are all smooth functions on* $\mathbb{R}^n \setminus \{0\}$ *that are all homogeneous of degree* $p \neq -1$*. Let*

$$g(x) = \frac{1}{p+1} \sum_{i=1}^{n} x_i f_i(x).$$

Prove that α *is exact, more precisely show that* $dg = \alpha$.

Problem 0.2. *Let* $\alpha = \sum_I a_I dx_I$ *and* $\beta = \sum_I b_I dx_I$ *be constant* k*-forms, i.e. with constant coefficients* a_I *and* b_I *(we also assume, as usual, that the multi-indices* I *are increasing). We define the inner product of* α *and* β *to be the number given by*

$$(\alpha, \beta) = \sum_I a_I b_I.$$

Prove the following assertions:

(i) The dx_I *form an orthonormal basis of the space of constant* k*-forms.*

(ii) $(\alpha, \alpha) \geq 0$ *for all* α *and* $(\alpha, \alpha) = 0$ *if and only if* $\alpha = 0$.

(iii) $\alpha(*\beta) = (\alpha, \beta) dx_1 dx_2 ... dx_n$.

(iv) $\alpha(*\beta) = \beta(*\alpha)$.

(v) The Hodge star operator is orthogonal, i.e. $(\alpha, \beta) = (*\alpha, *\beta)$.

Problem 0.3. *Let* U *be an open subset of* $\mathbb{R}^2$ *and let* $F = F_1 e_1 + F_2 e_2 : U \to \mathbb{R}^2$ *be a smooth vector filed. The differential form* β *defined by*

$$\beta = \frac{F_1 dF_2 - F_2 dF_1}{F_1^2 + F_2^2}$$

is well defined at all points $x \in U$ *where* $F(x) \neq 0$*. Let then* c *be a parametrized circle contained in* U*, traversed once in the counterclockwise direction. Assume that* $F(x) \neq 0$ *for all* $x \in c$*. The index of* F *relative to* c *is defined to be the number*

$$index(F, c) = \frac{1}{2\pi} \int_c \beta.$$

Prove the following assertions:

(i) $\beta = F^*\alpha$, where α is the angle form $(-ydx + xdy)/(x^2 + y^2)$.

(ii) β is closed.

(iii) $index(F, c)$ is the winding number of the curve $F \circ c$ about the origin and therefore is an integer.

Problem 0.4. *Show that for every disk in the plane of arbitrary center and radius, a* 1*-form defined on it is clossed, if and only if it is exact.*

Consider then an arbitrary circle in the plane, traversed once by a standard parametrization in the counterclockwise direction. Pick two points A and B, one in the interior of it and the other in its exterior. Calculate the winding numbers of this circle with respect to both A and B (recall that we learned how to calculate the winding number with respect to an arbitrary point, in one of the homework problems.)