MATH 4130
FINAL EXAM

STUDENT'S NAME:

PROBLEM 1:

PROBLEM 2:

PROBLEM 3:

PROBLEM 4:

PROBLEM 5:

PROBLEM 6:

TOTAL:

Problem 0.1. *Consider a convergent series of positive real numbers* $\sum_{n=1}^{\infty} a_n$ *and let* $\epsilon > 0$ *be fixed and arbitrarly small. Show that there exists another convergent series of positive real numbers* $\sum_{n=1}^{\infty} b_n$ *with* $a_n/b_n \to 0$ *and so that* $\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n + \epsilon$.

Problem 0.2. *Find the radius of convergence of the power series* $\sum_{j=0}^{\infty}(j+3)^2x^j$*. Then, find the function to which the series converges.*

Problem 0.3. *(1) Let f be a continuous function on the interval $[a, b]$. Suppose that for every rational numbers $\alpha, \beta \in [a, b]$ of the form $\frac{m}{2^n}$ (for some integers m and n) we know that*

$$\int_{\alpha}^{\beta} f(t)\, dt = 0.$$

Prove that $f(t) = 0$ for all $t \in [a, b]$.

(2) Let g be a continuous function on the interval $[a, b]$. Prove that there exists $c \in [a, b]$ so that

$$g(c) = \left(\frac{1}{b-a} \int_a^b (g(x))^{2025}\, dx \right)^{1/2025}.$$

Problem 0.4. *Consider $\Phi : [a,b] \to \mathbb{R}$ a C^2 function with the property that $\Phi''(x) \geq 0$ for every $x \in [a,b]$.*

(1) Show that for every $0 < \alpha, \beta < 1$ with $\alpha + \beta = 1$ one has

$$\Phi(\alpha x_1 + \beta x_2) \leq \alpha\Phi(x_1) + \beta\Phi(x_2)$$

for every $x_1, x_2 \in [a,b]$.

(2) Then, use the above fact to show that for every $x_1, x_2, ..., x_n \in [a,b]$ one has

$$\Phi(\frac{x_1 + x_2 + ... + x_n}{n}) \leq \frac{\Phi(x_1) + \Phi(x_2) + ... + \Phi(x_n)}{n}.$$

(3) Finally, use all of these to show that whenever f is continuous on the interval $[0,1]$ and $\Phi : \mathbb{R} \to \mathbb{R}$ is C^2 with $\Phi''(x) \geq 0$, then

$$\Phi(\int_0^1 f(x)dx) \leq \int_0^1 \Phi(f(x))dx.$$

Problem 0.5. *(1) Suppose that f is continuous and $|f(x)| < 3$ for all $x \in [a, b]$. Prove that*

$$\lim_{n\to\infty} \frac{1}{3^n} \int_a^b (f(t))^n \, dt = 0.$$

(2) Consider the function $\phi(x) = (h * h)(x)$ defined on the real line, where h is the characteristic function of the interval $[0, 1]$. Using it, define further the sequence of functions $\phi_n(x) := n\phi(nx)$ for every positive integer n. Prove that for every continuous and compactly supported function g, the sequence $g * \phi_n$ converges uniformly to g on the whole real line.

Problem 0.6. *Let f be a continuous and odd function (i.e., $f(x) = -f(-x)$) on the interval $[-1, 1]$. Show that if $\int_{-1}^{1} f(x)x^{2k+1}\,dx = 0$ for $k = 0, 1, 2, \ldots$ then f must be identically equal to zero.*