Equivariant Cohomology

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Contents

Introduction

Algebraically, Koszul duality can be thought of as an equivalence of two (co/)module categories for different rings. In the most classical setting, it is an equivalence between graded modules over the exterior algebra of a vector space and graded modules over the exterior algebra of its dual (plus some other adjectives). In this exposition we will describe a topological/geometric description of this duality in terms of group actions on spaces, and describe how, when taking (co/)homology, we arrive back at its algebraic form.

A remark on references: the bulk of this talk uses [1], though we try for a dg perspective as laid out in [5]. The expository note [2] were also useful and contains many examples, including that of the flag variety for SL_3 and the Grassmanian $\mathrm{Gr}(2,4)$. A more complete description and combinatorial model of the equivariant cohomology of Grassmanians is in [3], though we are unable to give much exposition here. References to some of the general theory of dg algebras is in [4] [5] [6]. Much of the theory is readily generalized to cohomology with values in some constructible sheaf as in [1] but we will not treat it in this note. All errors in these notes, which are sure to exist, are due to me. Corrections are very welcome.

Conventions

Everything is over an algebraically closed field of characteristic zero. Our grading conventions will be cohomological, i.e. differentials increase degree. Our topological spaces will always be "reasonable" in some sense – analytically constructible, or simplicial complexes. Throughout these notes let K be a compact Lie group; results may hold in greater generality but one should consult the references.

1 Two Koszul dual descriptions of equivariant cohomology on a space with a group action

1.1 Cohomology of sheaves over BK

Suppose we wanted to define something called "equivariant cohomology" using usual cohomology. For example, take the following naive definition.

Bad Definition. Let X be a topological space and K a topological group acting on X. Define the *equivariant cohomology*

$$H^\bullet_K(X) := H^\bullet(X/K).$$

This is bad because (1) if the action of K on X is not locally free, then X/K can be bad, e.g. non-Hausdorff, and (2) this construction does not seem to know much about the action of K (e.g. the difference between the adjoint action of K on itself and the trivial action of K on a point is not detected). These two defects hint at what should be the correct definition: we "replace X with a homotopy equivalent space with a free K-action."

For every topological group K, there is a homotopically unique contractible space, the *universal bundle* EK on which K acts freely. This can be constructed in a variety of ways: the Milnor construction, or delooping of simplicial groups, are two things that show up on nLab. We write EK/K by BK, the *classifying space*. Then, we define:

Definition 1 (Borel construction)**.** Let X be a topological space and K a topological group acting on X. Define the *equivariant cohomology* to be the graded $H^\bullet_K(\mathrm{pt}) = H^\bullet(BG)$-module

$$H^\bullet_K(X) := H^\bullet(X \times_K EK)$$

where $X \times_K EK = (X \times EK)/K$ under the diagonal action.

Example 1. If K acts freely on X, then $H^\bullet_K(X) = H^\bullet(X/K)$. In particular, for any group K, we have $H^\bullet_K(K) = \mathbb{C}$.

Example 2 (Action of a torus)**.** Let the Lie algebra $T = S^n \simeq (\mathbb{C}^*)^n$ be the compact real or complex torus. Then, $ET = (S^\infty)^n$ and $BT = (\mathbb{CP}^\infty)^n$. So $H^\bullet_T(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}] = S\mathfrak{t}^*$ with $\deg(\mathfrak{t}^*) = 2$.

Remark 1 (Sheaves over BK)**.** Normally, we can realize singular cohomology as the derived pushforward of the constant sheaf on X to a point (the terminal "base object" of the category we live in). In the K-equivariant setting, the terminal object should be BK, i.e. we are thinking about some sheaf on $X \times_K BK$, which we then push forward to BK.

I find it difficult to think of $H^\bullet_T(X)$ as a module over $H^\bullet_T(BK)$ geometrically, since BK tends to be a pretty big space for even "small" groups K. We have an equivalent description of equivariant cohomology using a kind of equivariant generalization of singular chains; we can get to equivariant cohomology by dualizing the resulting complex, but let us remain in homology for exposition's sake. We will sweep most details under the rug; a fuller exposition using subanalytic sets can be found in section 3 of [1].

Definition 2 (Equivariant chain complex)**.** Let $\dim(K) = k$. The ith dimensional K*-equivariant chains* $C_i^K(X)$ consist of K-equivariant maps $C \to X$, where C is a compact "reasonable" (possibly singular, subanalytic subsets of $\mathbb{R}^n$) $i + k$-dimensional space with a free K action. The boundary map is the usual boundary map.

Proposition 1. *The cohomology of the complex of K-equivariant cochains is isomorphic to equivariant cohomology defined above.*

Example 3. Let $T = S^1$. Then, the equivariant chains in $H_\bullet(\mathrm{pt})$ are given by compact spaces C with a locally free S^1 action (with with the only possible equivariant map $C \to \mathrm{pt}$). We can list them: there is a free S^1 action on S^{2n+1}; for $n = 0$ this is the regular action, for $n = 1$ this is the Hopf action. In general it is given by the unit length vectors of the action of $\mathbb{C}^* \simeq S^1$ on $\mathbb{C}^{n+1} = S^{2n+1}$.

1.2 Pointwise action on chains, i.e. a module over the homology algebra

We have the following Koszul dual description where we think about the action of K on points or chains in X. Chains on a space in general are a coalgebra with comultiplication induced by the diagonal map and counit induced by the unique map $K \to \mathrm{pt}$. If G also has a group action then $C_\bullet(K)$ is an algebra (in fact, a Hopf algebra) as follows. The unit and group multiplication on K give a map $K \times K \to K$, which induces an algebra structure on $C_\bullet(K)$, and the inverse on K is an antipode map. This allows us to define an action of $C_\bullet(K)$ on $C_\bullet(X)$ by the "sweep action," i.e. simply acting on the points of a chain by a chain in K.

Definition 3 (Sweep action)**.** Let $\xi \in C_j(X)$ and $s \in C_i(K)$; then the action on points of s on ξ gives us another subset, call it $s\xi$. If this is a $i + j$ dimensional chain, then define $s \cdot \xi = s\xi$. Otherwise, define $s \cdot \xi = 0$.

Example 4 (Action of a torus)**.** The homology ring of the 1-torus $T = (S^1)^n$ is $H_\bullet(T) = \Lambda\mathfrak{t}$, with $\deg(\mathfrak{t}) = -1$.

Remark 2 (Interpretation of trivial action)**.** Suppose there is a basis of $H_\bullet(X)$ consisting of K-closed chains. Then the action of $H_\bullet(G)$ on $H_\bullet(X)$ is trivial.

2 Koszul duality for dg algeberas and formal cohomological models

The above two descriptions of equivariant cohomology are related by Koszul duality, an equivalence between two module categories, with the caveat that when we pass from a chain complex (e.g. of cochains) to its cohomology, we may lose some important data that we should have remembered. This Koszul duality interchanges free and trivial modules. The first two subsections here give the necessary background and may be a bit dry.

2.1 DG Algebras

We will first give a brief introduction to differential graded algebras, which one can think of as a natural algebra object one would define in the category of complexes of vector spaces, with the below specified monoidal (tensor) product.

Definition 4. A *dg algebra* is a chain complex $C^\bullet$ with a multiplication map $m : C^\bullet \otimes C^\bullet \to C^\bullet$ which commutes with the differential. The real data is the definition of the differential on the tensor product $d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db)$, given by the Leibniz rule. A dg algebra is *dg commutative* if m commutes with switching the factors in $C^\bullet \otimes C^\bullet$, i.e. $ab = (-1)^{|a|} ba$. Given a dg algebra A, the associated *cohomology ring* $H^\bullet(A)$ is the dg algebra with trivial differentials given by taking cohomology of the complex.

Example 5. Singular cohomology is a dg algebra, with multiplication given by the cup product. This dg algebra is not commutative, but dg commutative.

Remark 3 (Free dg algebras)**.** In the category of commutative k-algebras, we have a construction for the universal free algebra on a set of generators S, i.e. the polynomial ring $k[S]$. In the category of commutative dg-k-algebras, we have a similar construction, the difference that (1) we must specify the degree of each element of S and (2) we may also specify the differentials of generators S on homogeneous-degree words in S with coefficients in k. Generators in odd degree behave like exterior variables, and generators in even degrees behave like symmetric variables, and differentials are defined on words by the dg Leibniz rule.

Remark 4 (Derived categories and Verdier quotients)**.** We have a notion of derived categories for module categories of dg-algebras. A construction of Drinfied [4] realizes Verdier quotients in the dg setting as follows: let $\mathcal{C}$ be a dg-category and $\mathcal{D}$ a full subcategory. For objects $X \in \mathcal{D}$ we want to "kill," we freely adjoin $\mathrm{Hom}_{\mathcal{C}/\mathcal{D}}(X, X) = \mathrm{Hom}_{\mathcal{C}}(X, X)[\eta]$ with $\deg(\eta) = -1$ and $d(\eta) = \mathrm{id}_X$. One can use this construction to localize categories: we quotient by the full subcategory whose objects consist of mapping cones of morphisms we wish to invert.

Example 6. In light of the above remark, we can write, with $\deg(\beta) = 2$ and $\deg(\lambda) = -1$,

$$H_\bullet(S^1) = k[\lambda],$$

$$H^\bullet_{S^1}(\mathrm{pt}) = k[\beta],$$

where λ is an exterior variable and β a symmetric variable. Note that since complexes are cohomologically graded, we put homology in negative degrees.

Remark 5 (Free dg module and dual free module)**.** Let A be a dg algebra. As usual there is a free left A-module which we write $A^\wedge$ (as in [6]). There is also a dual free module

$$A^\vee = \mathrm{Hom}_k(A^\wedge, k)$$

where we treat $A^\wedge$ as a right A-module so that $A^\vee$ is a left A-module. This will show up in a dg Koszul duality.

Remark 6 (Dg categories)**.** The full description of dg categories would be something like this. Let $\mathcal{C}$ be a monoidal category, i.e. a category with a tensor product on objects satisfying some coherence axioms. In general we say a category is *enriched* over $\mathcal{C}$ if the Hom-sets of that category are objects in $\mathcal{C}$, and composition morphisms in $\mathcal{C}$. An algebra enriched over $\mathcal{C}$ is such a category with a single object. Given such a category, the category of *left modules* is a functor from that category to $\mathcal{C}$. The category of *right modules* is a functor from the opposite category.

If $\mathcal{C}$ is the category of k-modules, then a category enriched over $\mathcal{C}$ is said to be k-linear. If $\mathcal{C}$ is the category of chain complexes over k with the above specific monoidal structure (satisfying the Leibniz rule), then categories enriched over $\mathcal{C}$ are called k-linear dg-categories.

Example 7. Let A be a commutative ring and $\mathbf{C}_{\text{dg}}(A)$ the dg category of whose objects are complexes of A-modules and whose Hom-sets are the graded complex all graded maps of any degree (which do not have to respect the differential), with differential $f \mapsto fd - (-1)^{|f|}df$. This is a useful notion; one can show that $H^i(\text{Hom}_{\mathbf{C}_{\text{dg}}(A)}(X, Y)) = \text{Hom}_{\mathbf{K}(A)}(X, Y[i])$.

2.2 Formal minimal A_∞ models for a dg algebra

A dg algebra may fail to be quasi-isomorphic to its cohomology algebra. The standard topological example is the knot theory of the Borromean triple-linked rings. Likewise, a module over a dg algebra can also fail to be quasi-isomorphic to its cohomology module. We will give many examples later; here we will introduce some words which we won't explain in detail. The interested reader may refer to [6].

Definition 5. An A_∞ *algebra* is a graded algebra with a first-order multiplication of degree 1 (i.e. a differential), a second-order multiplication of degree 0 (i.e. the multiplication of a usual algebra), and higher order multiplications of order n of degree $2 - n$. There is a notion of *morphisms* of A_∞ algebras; the morphisms are a collection of maps f_n of degree $1 - n$ satisfying some coherence relations. We say two A_∞-algebras are *quasi-isomorphic* if there is a morphism between them, and when taking cohomology the morphism of k-modules f_1 becomes an isomorphism of algebras. It turns out that *every quasi-isomorphism is a homotopy equivalence* (for some appropriate notion of homotopy equivalence). There is also, similarly, a notion of A_∞-modules.

Remark 7 (Dg algebras are A_∞ algebras (and vice versa))**.** A dg algebra can be thought of as an A_∞ algebra by simply letting the higher multiplications be zero. Under certain conditions every A_∞-algebra is quasi-isomorphic to a dg algebra; this is not obvious (c.f. [6]).

Definition 6. Let A be a dg algebra. We will say that an A_∞ algebra $H(A)$ is a *formal minimal A_n-model* for A if it is quasi-isomorphic to A and has trivial differential (i.e. 1-multiplication) and trivial i-multiplications for $i > n$. Note that n can be infinity. By necessity $H(A)$ is the cohomology ring of A.

Example 8 (Every chain complex of vector spaces has an A_2 formal model)**.** This is essentially due to the fact that every short exact sequence splits. Let V be a vector space with subspaces $B \subset Z \subset V$. We can write $Z = B \oplus B^\perp$ and $V = Z \oplus Z^\perp$ so that $V = B \oplus B^\perp \oplus Z^\perp$. Then there is a map $Z/B \to V$ which is given by the isomorphism $Z/B \cong B^\perp$ postcomposed with inclusion. This argument is applied to every level of the chain complex and its cohomology.

Proposition 2. *Every dg algebra has a formal A_∞ model, i.e. is quasi-isomorphic to its cohomology as an A_∞-algebra where the first-order multiplication is zero.*

Remark 8. In other words, there is a morphism $f : H(A) \to HA$ which is a quasi-isomorphism. One can think of the map f_1 as "choosing representatives for cohomology classes," i.e. for $x \in H(A)$, $f_1(x)$ is its chosen representative. It may turn out that $f_1(x)f_1(y) - f_1(xy)$ is nonzero; however, it should be killed by a coboundary. We then define the morphism $f_2(x, y)$ to choose such a coboundary. Going further, f_3 enforces the associativity law, except now we may have to define a multiplication m_3 on $H(A)$ to make the coherence relation work

$$f_1 \circ m_3 + f_2 \circ (m_2 \otimes 1 - 1 \otimes m_2) = m_2 \circ (f_1 \otimes f_2 - f_2 \otimes f_1) + m_1 \circ f_3$$

Before we go into this coherence relation, let us talk about Massey products, i.e. try to choose f_3, m_3 when $(m_2 \otimes 1 - 1 \otimes m_2) = 0$ above.

Definition 7 (Massey products)**.** Let A be a dg algebra. Suppose that $[a][b] = 0$ and $[b][c] = 0$. Define the *Massey product* $\langle [a], [b], [c] \rangle = [sc + at]$, for some s, t with $d(s) = ab$ and $d(t) = bc$. (Note: there are signs on these terms which I am entirely ignoring.) The Massey product may be undefined in the sense that there may be more than one choice.

Now, let us write this in terms of our above notation. Suppose the Massey product vanishes. Let $f_3([a],[b],[c])$ be a coboundary that kills $sc + at$ and write $s = f_2([a],[b])$ and $t = f_2([b],[c])$. Let $f_1([a]) = a$ and similarly. Then we have

$$(af_2(b,c) - f_2(a,b)c) + df_3(a,b,c) = 0$$

i.e. one can make choices such that $m_3([a],[b],[c]) = 0$. The existence of a formal A_∞-model for A where multiplications for $n \geqslant 3$ vanish implies that all Massey products vanish. However, the converse is untrue; roughly, one must have that Massey products vanish "in a uniform way" (uniform with respect to the choices we made when we defined f_2).

Remark 9. Essentially, A_∞ algebras give us another way to repackage the information in a dg algebra so that we have no boundary maps at the cost of having higher multiplications.

One should refer to section 13 of [1] and also [6] for a more complete description, as well as connections to cyclic homology.

2.3 Koszul duality

We now focus on the specific case when $T = (\mathbb{C}^\bullet)^n$. Define $S := H_T^\bullet(*) = S\mathfrak{t}$ and $\Lambda := H_\bullet(T) = \Lambda\mathfrak{t}^*$, where $\deg(\mathfrak{t}) = -1$ and $\deg(\mathfrak{t}^*) = 2$. We use the conventions as in [5]; a notable difference in [1] is that their explicit definition has both $\mathfrak{t}$ and $\mathfrak{t}^*$ in positive degree.

Theorem 1 (Koszul duality for dg algebras)**.** *There is a derived equivalence*

$$\mathbf{D}(S^{op}) \underset{-\otimes^L_\Lambda k}{\overset{R\operatorname{Hom}_S(k,-)}{\rightleftarrows}} \mathbf{D}(\Lambda).$$

Furthermore, these equivalences identify trivial objects on the left with free objects on the right, and free dual objects on the left with trivial objects on the right.

$$k \leftrightarrow \Lambda$$

$$S^\vee \leftrightarrow k$$

Taking the full triangulated subcategory, we have an equivalence:

$$\mathrm{DPerf}(S^{op}) \leftrightarrow \mathrm{DFinDim}(\Lambda).$$

Dually, there is also a derived equivalence

$$\mathbf{D}(S^{op}) \underset{R\operatorname{Hom}_\Lambda(k,-)}{\overset{-\otimes^L_S k}{\rightleftarrows}} \mathbf{D}(\Lambda).$$

Furthermore, this equivalence identifies free objects on the left with trivial objects on the right, and trivial on the left with free dual objects on the right.

$$k \leftrightarrow \Lambda^\vee$$

$$S \leftrightarrow k$$

Taking the full triangulated subcategory, we have an equivalence:

$$\mathrm{DFinDim}(S^{op}) \leftrightarrow \mathrm{DPerf}(\Lambda).$$

Proof. The proof is essentially by some dg version of Morita theory/Barr-Beck formalism. The functors $R\operatorname{Hom}(k,-)$ and $-\otimes k$ receive a right-action of $R\operatorname{Hom}(k,k)$. The construction of these adjoint functors is verified by the computation that $R\operatorname{Hom}_S(k,k) \simeq \Lambda$ and $R\operatorname{Hom}_\Lambda(k,k) \simeq S$. One uses the Koszul resolution: $k[\lambda][\beta^*] \to k$ where $d(\beta^*) = \lambda$ ($\deg(\beta^*) = -2$ and $k[\beta][\lambda^*] \to k$ where $d(\lambda^*) = \beta$ ($\deg(\lambda^*) = 1$). That it is an equivalence follows essentially from the formalism of [5]; for an augmented k-dg-algebra, k is a compact generator (i.e. the smallest full triangulated subcategory closed under infinite coproducts is the entire category). ☐

Remark 10 (Opposite ring)**.** Since S is commutative, $S = S^{\mathrm{op}}$ and we will suppress this in the notation.

Remark 11 (Dual free object)**.** Λ is a Frobenius algebra; what this means in a graded setting is that $\Lambda^\vee \cong \Lambda[n]$. Thus if we use the second derived equivalence, free objects are identified with trivial objects on both sides. However, the same is not true for the first derived equivalence.

The following will be useful in identifying equivariant cohomology with usual cohomology. The proof is by standard adjunctions.

Proposition 3. *The forgetful functor intertwines with extension of scalars to k, i.e. let F be the forgetful functor. If M is a S-module and N is a Λ-module, and M and N are Koszul dual, then $k \otimes_S^L M = \mathrm{Hom}_\Lambda(k, N)$ and $k \otimes_\Lambda^L N = \mathrm{Hom}_S(k, M)$.*

2.4 Failure of Koszul duality in cohomology

Up to this point, we haven't yet shown that Koszul duality actually corresponds with the two notions of equivariant cohomology defined earlier. In fact, this is strictly speaking not true – the reason being that cohomology is not always a formal model for a cochain complex without some extra A_∞-structure. Here we will show this failure by example and find conditions for when cohomology is a formal A_2-model.

Remark 12. One can readily see that $H_\bullet(T)$ is a formal model for $C_\bullet(T)$ and that $H^\bullet(BT)$ is a formal model for $C^\bullet(BT)$.

The following two examples will illustrate a failure of Koszul duality in cohomology; two spaces with different actions detected by the cohomology side but not detected by the homology side.

Example 9 (The trivial S^1 action on S^3)**.** If S^1 acts on X trivially, then $H_T^\bullet(X) = (\mathbb{C} \oplus \mathbb{C}[3]) \otimes \mathbb{C}[\beta]$. On the dual side of things, $H_\bullet(T)$ acts on $H_\bullet(X)$ trivially for degree reasons.

Example 10 (The Hopf action of S^1 on S^3 in cohomology)**.** The Hopf action of S^1 on S^3 is free with quotient $S^3/S^1 = S^2$. Then $H_T^\bullet(X) = \mathbb{C} \oplus \mathbb{C}[-2] = \mathbb{C}[\beta]/\beta^2$ with $\deg(\beta) = 2$; one determines the $\mathbb{C}[\beta]$-action by thinking about the map $X \times \mathrm{pt} \to X$ and the cochains $[S^1] \times_{S^1} [S^3] \mapsto [S^3]$. Dually, $H_\bullet(T)$ acts on $H_\bullet(X)$ trivially also for degree reasons.

As expected, the problem is of formality.

Example 11 (The trivial action of S^1 on S^3 is formal)**.** The trivial action of $C_\bullet(S^1)$ on $C_\bullet(S^3)$ is identically zero by definition. Thus, a quasi-isomorphism of $C_\bullet(S^1)$-modules $C_\bullet(S^3) \cong H_\bullet(S^3)$ is simply an isomorphism of $\mathbb{C}$-vector spaces, which always exists.

Example 12 (The Hopf action of S^1 on S^3 on cochains)**.** Now consider the Hopf action. We claim that it is not formal. Indeed, there is a 2-chain in S^3 which sweeps by the S^1 action to the a representative of the fundamental class. Thus, no map between $C_\bullet(S^3)$ and $H_\bullet(S^3)$ can be a quasi-isomorphism of $C_\bullet(S^1)$-modules. In fact, one can check that taking the Koszul dual of the $k[\beta]$-module $k[\beta]/\beta^2$ yields a dg-model for $C_\bullet(S^3)$ which is not quasi-isomorphic to its cohomology.

However, if we give $H_\bullet(S^3)$ an A_∞ structure, a difference is detected. In fact, it is exactly the existence of these higher cohomological operations that obstruct a quasi-isomorphism to cohomology.

Example 13 (Higher cohomological operations of S^1 on S^3)**.** Take a point and sweep it by S^1. The result is homologous to zero by a the boundary of a 2-chain. The sweep action of S^1 on this 2-chain is the fundamental 3-cycle. So we could say that the 3-action of the class $[S^1]$ on $[\mathrm{pt}] \in H_\bullet(S^3)$ is $[S^3]$.

For good measure, we will provide an example where the $H_\bullet(S^1)$ acton is not trivial but still formal. The dual $H_{S^1}^\bullet(\mathrm{pt})$ action is not free.

Example 14 (Three 2-spheres in a ring)**.** Let X be three 2 spheres glued toegher at 0 and ∞ to form a ring, and let S^1 act on each sphere by rotation fixing $0, \infty$. One checks that $H_\bullet(X) = \mathbb{C} \oplus \mathbb{C}[1] \oplus \mathbb{C}[3]^{\oplus 3}$, and the S^1 action takes the 0-cycle to zero and the 1-cycle to the sum of the 3-cycles, i.e. the action is not trivial. One can check that it is indeed formal – the 2-chain killing off the action of S^1 on the 0-cycle is killed by the S^1 action, so higher cohomological operations vanish.

However, the "fundamental class" is a chain in $H^1_T(X)$, but $H^3_T(X)$ is trivial, so the action is not free. I do not know whether the module is formal or not.

We now state our promised result.

Proposition 4. *The following are equivalent.*
(1) $H_\bullet(X)$ is a formal (A_2-)model and trivial over Λ
(2) $H^\bullet_T(X)$ is a formal (A_2-)model and free over S
(3) The double complex spectral sequence associated with computing $k \otimes^L_S -$ collapses at E_1
(4) The Eilenberg-Moore spectral sequence associated to the fibration $X \to X \times_T ET \to BT$ collapses at E_2
(5) Λ acts trivially on $H_\bullet(X)$ and *the "higher cohomology operations" of Λ vanish, i.e. the minimal A_∞ model for $H_\bullet(X)$ has trivial higher multiplications.*

Proof. For (1) and (2), it is a general fact of Koszul duality that $R\,\mathrm{Hom}(k, -)$ takes trivial objects to free objects and $k \otimes^L -$ takes free objects to trivial objects. Formality just means we are not *really* dealing with complexes, so this result holds in the derived setting as well. (5) is easily seen to be equivalent to (1). The argument for (3) and (4) can be found in [1]. □

Remark 13. Note that it is not true that $H_\bullet(X)$ is formal if and only if $H^\bullet_T(X)$ is formal! Many examples above.

Finally, we have the actual theorem which tells us that there is a Koszul duality relationship between these two descriptions of equivariant cohomology. Its statement, at least in the form I have seen it, is rather technical so I would simply like to do this:

Theorem 2 ([1] Theorem 11.2)**.** *There are functors*

$$G : \mathbf{D}^b_{T,c} \to \mathbf{D}^f_+(H^\bullet(BK)))$$

$$E : \mathbf{D}^b_{T,c} \to \mathbf{D}^f_+(H_\bullet(K))$$

These functors are equivalences of categories and are related by Koszul duality.

As a closing remark, note that we should have this kind of Koszul duality for any compact Lie group; see [7] for a discussion and references.

Proposition 5. *Let K be a compact Lie group. Then $H^\bullet(BK)$ is a free symmetric algebra, and $H_\bullet(K)$ is a free exterior algebra.*

3 The localization theorem, computing equivariant cohomology, examples

In this section we introduce the localization theorem, a nice tool for helping us compute equivariant cohomology. The conditions of the theorem call for a free $H^\bullet_T(\mathrm{pt})$-module; the just-stated proposition gives us another way to view this condition.

3.1 The localization theorem

The following theorem is the main theorem of [1]. I have not looked carefully at the proof, but I include the lemma that seems like it is somewhat key.

Theorem 3 (Localization theorem)**.** *Suppose that $H_T^\bullet(X)$ is a free module over $H_T^\bullet(\mathrm{pt})$. Let X_i be the set of $\leqslant i$ dimensional orbits, and assume each has only finitely many strata. Then the sequence*

$$0 \to H_K^\bullet(X) \to H_K^\bullet(X_0) \to H_K^\bullet(X_1, X_0)$$

is exact.

Lemma 1. *Let $Y \subset X$ be a single orbit of positive dimension. Then $H_K^\bullet(Y)$ is a torsion module over $H_K^\bullet(\mathrm{pt})$.*

Proof. The rough idea is that, let S be the stabilizer of the orbit, and let $T = S \times P$. Then we should have P acting (almost?) freely on Y, so one can check that the corresponding subalgebra $S\mathfrak{p}^* \subset \mathfrak{t}^*$ should act by torsion. □

We now state some sufficient conditions. Here we copy part of the list in [1] of sufficient conditions for equivariant formality, omitting the ones that don't make sense in this note due to simplifications chosen.

Proposition 6. *Let X be a space with an action of K. The following are sufficient conditions for a cohomology to be equivariantly formal.*
(1) Cohomology vanishes in odd degree.
(2) The Λ-action on homology is trivial and all higher operations vanish.
(3) The homology groups are generated by K-invariant subanalytic cycles.
(4) X can be decomposed into K-invariant cells.
(5) X is a smooth complex projective variety, and $K = (S^1)^r$ is the compact form of $T = (\mathbb{C}^*)^r$.
(6) X is a compact symplectic manifold, and K acts on X be Hamiltonian vector fields.
(7) The homology of X vanishes in odd degrees, and $H_\bullet(X)$ has a basis of cycles which are closed under the action of K.

All of our examples will be smooth projective complex algebraic varieties; in this case homology vanishes in odd degrees, so all higher cohomology operations mush vanish by necessity of degrees, satisfying (7).

3.2 The data of a space using the localization theorem

The localization theorem tells us that we can compute the equivariant cohomology of a space by looking at data associated to its fixed points and one-dimensional orbits by carefully analyzing the maps in the theorem. We can encode this data in a graph. Let $T = (\mathbb{C}^*)^n$; in particular, T is abelian.

1. We want to identify $H_T^\bullet(X)$ with a submodule of $H_T^\bullet(X_0) = \bigoplus_{x \in X_0} S\mathfrak{t}^*$. So we should first lay out this ambient object $H_T^\bullet(X_0)$: for each T-fixed point in X, draw a node, which will represent a summand of $H_T^\bullet(\mathrm{pt})$ corresponding to that fixed point.

2. Every one-dimensional orbit O is canonically (since T is abelian it does not matter which point we choose to compute the stabilizer) isomorphic to $T/\mathrm{Stab}(O) \cong \mathbb{C}^*$ and has two fixed points in its closure. In particular, we note that $H_T^\bullet(O) = \mathbb{C}[\mathfrak{s}] = S\mathfrak{s}^\perp$. where $\mathfrak{s}$ is the Lie algebra of $\mathrm{Stab}(O)$. For each orbit, draw an edge between these two nodes. By choosing a notion of positive roots of T we can assign each edge a direction. This gives us a partial ordering on the nodes. One way to depict this is to let the "northern" nodes be more positive.

3. The map $H_T^\bullet(X_0) \to H_T^\bullet(X_1, X_0)$ is the standard coboundary map. We can think of it locally; let $F = \{s, t\}$ be a set of fixed points in the closure of a single one-dimensional orbit $O \cong \mathbb{C}^*$, where s is a source and t is a target (sink). Then we have a map $H_T^\bullet(F) = k[\mathfrak{t}] \otimes \{s, t\} \to k[\mathfrak{s}] = H_T^\bullet(O)$. Explicitly, this map sends $f_s s + f_t t \mapsto f_s - f_t$ for $f_s, f_t \in k[\mathfrak{t}] = S\mathfrak{t}^*$; we want $f_s - f_t$ to be zero on the stabilizer $\mathfrak{s}$, i.e. $f_s - f_t \in k[\mathfrak{t}/\mathfrak{s}] = S\mathfrak{s}^\perp$. This is a generated by a single nonzero vector in $\mathfrak{t}^*$. Label the edge with that vector.

4. The game is now to assign polynomials in $S\mathfrak{t}^*$ to the nodes that satisfy the conditions. From this we can determine a basis of the cohomology ring.

Example 15 (The simplest example)**.** The simplest example of a torus action is the usual $\mathbb{C}^*$-action on $\mathbb{CP}^1$. Here we have two nodes corresponding to points $\{N, S\}$ and a single edge, labeled by t, since there is no stabilizer. Thus condition is that $t \mid p_N - p_S$. We can find a basis by setting $p_S = p_0$ and then $p_N = p_0 + tp_1$; the basis is $(p_S, p_N) = (1,1), (0,t)$. Thus $H^\bullet_{S^1}(\mathbb{CP}^1) = \mathbb{C}[\beta] \otimes \mathbb{C}[\beta]/\beta^2 = H^\bullet_{S^1}(\mathrm{pt}) \otimes H^\bullet(\mathbb{CP}^1)$. As a side remark, the 0-cycle is the inclusion of S^1 into an orbit, and the 2-cycle is the Hopf fibration map.

3.3 Example: flag varieties

The following two examples will only be sketched. They are probably best left as exercises, though the following exposition might be useful if the reader is ever stuck. The reader should refer to [2] and [3] for some nice graphs depicting the answers, though I use some different conventions which I lay out here. I will draw some graphs and scan it in the handwritten notes that accompany my talk, since I don't really want to spend forever messing around with xymatrix.

Let G be a semisimple complex Lie group B a Borel and $T \subset B$ a maximal torus. Let the Weyl group be $W_T = N(T)/T$. Then we have the Bruhat decomposition $G = BWB$ and also the opposite Bruhat decomposition $G = B^{\mathrm{op}}WB$. We will use the former convention. The flag variety is G/B, and the fixed points are in bijection with W_T. Further we know that the usual cohomology is generated by the Schubert cells, i.e. the closures of $BwB/B \subset G/B$.

We will explicitly do out the equivariant cohomology of the flag variety $X = B/G$ with $G = SL_3$ and B the upper triangular matrices. We take T to be the diagonal matrices. Let $\mathfrak{t}^*$ have the basis:

$$t_1 \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} = h_1 + h_2 - 2h_3 \qquad t_2 \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} = h_2 + h_3 - 2h_1.$$

One can check that the fixed points are the permutation matrices, directly or by noting that they are representatives of $N(T)B/B$. The one-dimensional orbits are given by matrices that look like: $\begin{pmatrix} 0 & 0 & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and *not* like: $\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, i.e. the parameter should not be to the right of any 1. Taking $a \to 0$ we find one fixed point and $a \to \infty$ we find the other, which is obtained by taking $a = 0$ and swapping the rows as in the following example:

$$\lim_{a \to \infty} \begin{pmatrix} 0 & 0 & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We already chose the positive (or negative? I do not really care as long as they are consistent) roots. Let the identity matrix be on the bottom. An edge labeled with t_1 will swap the first and second rows, an edge labeled with t_2 will swap the second and third rows, and an edge labeled with $t_1 + t_2$ will swap the first and third rows. The equivariant Schubert cells can be determined as follows. Choose a node; all nodes below it in the order will be nonzero, and all other nodes zero. This forces some conditions, which we will see by example.

3.4 Example: Grassmanians

We will use the following conventions: we want to represent a cell as a 2×4 matrix in reduced upper row echelon form, i.e. the top-dimensional cell is

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$

and the lowest-dimensional cell is

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The combinatorial data we want to extra from a cell is the intersection with some transverse flag – we will take the reverse flag

$$0 \subset \langle e_4 \rangle \subset \langle e_3, e_4 \rangle \subset \langle e_2, e_3, e_4 \rangle \subset \mathbb{C}^4$$

and take the dimensions of the intersections with the cells. For example, in the top-dimensional cell we have 00012 and for the low-dimensional cell we have 01222. We can re-hash this information by taking the differences between successive integers to get a string of 0s and 1s, for example 0011 and 1100. These strings are recording the columns that the 1s are in, reading from the right. The cells are in bijection with strings of length four with two 1s.

The action of the torus $T = (\mathbb{C}^*)^4$ scales the columns of the matrix. Take the basis t_1, t_2, t_3, t_4 of $\mathfrak{t}^*$, where t_i is dual to the vector which corresponds to scaling the ith column. This action has a stabilizer which is the diagonal scalar action, so really we have a torus action of $T = (\mathbb{C}^*)^4/(1,1,1,1)\mathbb{C}^* \cong (\mathbb{C}^*)^3$. However, the above basis is still convenient – quotienting out by $(1,1,1,1)$ means we should always have that the sums of the coefficients of the t_i be zero. The fixed points are the matrices above without the stars, e.g.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The one-dimensional orbits are obtained by inserting any a in a column which does not have a 1 as follows, and with limit switching the positions of 1 and a in the row:

$$\lim_{a \to \infty} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We can fix a sign convention by asking that the a's be inserted only to the right of the 1s. Using the strings of 0s and 1s, the one-dimensional orbits correspond to transpositions which change the string, i.e. swapping the first and second positions in 1100 is not a one-dimensional orbit. The transposition (ij) corresponds to $t_i - t_j$ with some sign convention. This should yield the diagram depicted in [3].

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