Almost complete intersections and the Lex-Plus-Powers Conjecture

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Abstract

We prove the almost complete intersection case of the Lex-Plus-Powers Conjecture on graded Betti numbers. We show that the resolution of a lex-plus-powers almost complete intersection provides an upper bound for the graded Betti numbers of any other ideal with regular sequence in the same degrees and the same Hilbert function. A key ingredient is finding an explicit comparison map between two Koszul complexes. Finally, we obtain bounds on the Hilbert function of an almost complete intersection, including a special case of a conjecture of Eisenbud-Green-Harris.

1 Introduction

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k. There is a close relationship between the Hilbert functions of homogeneous ideals I in R and their graded free resolutions. Given the graded free resolution of R/I, one can determine its Hilbert function $H_{R/I}$ by using the graded Betti numbers to determine the rational function expression of the Hilbert series. However, there may be many ideals with a given Hilbert function that have different minimal graded free resolutions. There has been a considerable effort in the last decade to determine what graded free resolutions can actually occur for a given Hilbert function.

To study this problem, it is natural to impose a partial order on the resolutions of ideals with the same Hilbert function. Let I and J be homogeneous ideals in R with the same Hilbert function. Then we say that $\beta^{R/I} \leq \beta^{R/J}$ if and only if $\beta_{ij}^{R/I} \leq \beta_{ij}^{R/J}$ for all i and j. This is a strong condition; the inequality has to go the same way for every graded Betti number. In particular, it is not clear *a priori* that there should be a unique maximal or minimal resolution for a given Hilbert function. In fact, Charalambous and Evans have shown that there may be incomparably minimal resolutions for a single Hilbert function [3] (see also the end of [12]). Richert [17] and Rodriguez [19] have done closely related work as well.

However, the situation is better at the top of the partial order. Bigatti and Hulett independently proved the main result about largest Betti numbers in characteristic zero, and Pardue generalized it to positive characteristic. Recall that a lexicographic ideal is a monomial ideal generated in each degree by an initial segment of monomials in descending lexicographic order. Macaulay showed that lexicographic ideals can attain the maximal possible Hilbert function growth [14]. The following result from the early 1990s of Bigatti [1], Hulett [12], and Pardue [16] establishes the maximality of the graded Betti numbers of lexicographic ideals.

Theorem 1.1 *Let $L \subset R$ be a lexicographic ideal, and let $I \subset R$ be a homogeneous ideal such that $H_{R/I} = H_{R/L}$. Then $\beta^{R/I} \leq \beta^{R/L}$.*

Consequently, the search for all the resolutions that occur for a Hilbert function is a bounded problem. That is, we need only consider graded free resolutions that lie below that of the lexicographic ideal, so there are only finitely many possibilities. Unfortunately, it is still relatively difficult to show that a particular candidate for a resolution cannot occur; see [6] for some techniques.

One would like to find ideals with analogous properties to those of the lexicographic ideals in more restrictive settings to try to get more information about the possible Hilbert functions and graded free resolutions that can occur. A logical place to start is the case in which one considers ideals that contain a regular sequence of maximal length in prescribed degrees. In [5], Eisenbud, Green, and Harris identify a candidate for maximal Hilbert function growth in this setting.

Conjecture 1.2 *Fix a nonnegative integer d. Let I be a homogeneous ideal containing a maximal length regular sequence in degrees $a_1, \ldots, a_n$. Suppose we can form a monomial ideal L with minimal generators $x_1^{a_1}, \ldots, x_n^{a_n}$ plus the first l monomials in degree d in descending lexicographic order, where l is selected so that $H_{R/I}(d) = H_{R/L}(d)$. Then $H_{R/I}(d+1) \leq H_{R/L}(d+1)$.*

The most interesting case is when I contains a regular sequence of length n in degrees $a_1, \ldots, a_n$ and no smaller degrees; then we can always form the ideal L described above. This conjecture is known for I a monomial ideal; it follows from the theorem of Clements and Lindström [4]. There has been some progress when the a_i are all 2; see [11] and [8]. Eisenbud, Green, and Harris were interested in the conjecture because they show in [5] that a special case of it implies their Generalized Cayley-Bacharach Conjecture in algebraic geometry.

The ideals L in the conjecture are part of a larger class of ideals whose graded Betti numbers we wish to consider. They are the natural analogues of lexicographic ideals in the situation in which we require our ideals to contain a regular sequence of maximal length in prescribed degrees.

Definition 1.3 *Let $a_1 \leq \cdots \leq a_n$ be positive integers. We call L an $(a_1, \ldots, a_n)$ $-$lex-plus-powers (LPP) ideal if:*

(1) L is minimally generated by $x_1^{a_1}, \ldots, x_n^{a_n}$ *and monomials* $m_1, \ldots, m_l$*, and*
(2) If r *is a monomial,* $\deg r = \deg m_i$*, and* $r >_{lex} m_i$*, then* $r \in L$*.*

For example, $L = (x_1^2, x_2^3, x_3^3, x_1x_2^2, x_1x_2x_3)$ is a $(2,3,3)-$LPP ideal. It contains appropriate powers of the variables, and we need only check the second condition for the other two generators. Since x_1^3, $x_1^2x_2$, and $x_1^2x_3$ are all in L, L is an LPP ideal. One builds the ideal by first forming the regular sequence of maximal length and then adding more generators in descending lexicographic order to get the desired Hilbert function.

Intuitively, if we believe Conjecture 1.2, the correspondence between maximality of Hilbert function growth and graded Betti numbers of lexicographic ideals should carry over to this new setting. This relationship led to the formulation of the LPP Conjecture (whose history is murky but is perhaps best described as due to Evans and inspired by Eisenbud-Green-Harris) in a paper of Evans and Richert [6]:

Conjecture 1.4 (LPP Conjecture) *Let* $L \subset R$ *be an* $(a_1, \ldots, a_n)-$*LPP ideal. Suppose* $I \subset R$ *is a homogeneous ideal with the same Hilbert function that contains a regular sequence in degrees* $a_1, \ldots, a_n$*. Then* $\beta^{R/I} \leq \beta^{R/L}$*.*

The LPP Conjecture is completely natural given our knowledge of lexicographic ideals and Conjecture 1.2. There is substantial computational evidence for it, but proving the conjecture in its full generality seems difficult. One is tempted to borrow from the proofs of Bigatti and Hulett in the lexicographic case; for example, instead of comparing any ideal I to a lexicographic ideal, they consider the generic initial ideal of I. In characteristic zero, this gives a strongly stable ideal with graded Betti numbers the same or larger than those of I, and one has convenient formulas for the graded Betti numbers of strongly stable ideals. However, if we were to do something similar, we would wish not only to keep the Hilbert function the same, but we would need also to fix the degrees of the regular sequence. Thus the generic initial ideal poses problems. Moreover, even though Charalambous and Evans have found the minimal resolution for LPP ideals [2], it can be hard to use it to compare LPP ideals to other ideals, partially because of some unpredictable ideal quotients that arise.

The case of the LPP Conjecture in which the LPP ideal is a complete intersection is trivial; the first nontrivial case is when the LPP ideal is an almost complete intersection. We prove this in Theorem 6.1:

Theorem 6.1 *Let* L *be an* $(a_1, \ldots, a_n)-$*LPP almost complete intersection. Let* I *be any ideal with the same Hilbert function as* L *that contains a regular sequence in degrees* $a_1, \ldots, a_n$*. Then* $\beta^{R/I} \leq \beta^{R/L}$*.*

We adopt an approach different from that of Charalambous and Evans to resolve almost complete intersection ideals. Instead of aiming for minimal free resolutions at the start, we form nonminimal resolutions and then try systematically to detect the nonminimality.

Our main interest in this paper is the LPP Conjecture and related questions about Hilbert functions. We note that Migliore and Miró-Roig have done substantial work in a different direction in [15] to find the minimal graded free resolution of almost complete intersections whose generators are generic. Our results give sharp upper bounds for the graded Betti numbers of any ideal with the same Hilbert function as an LPP almost complete intersection, not only generic $(n+1)$-generated ideals. However, the disadvantage is that many almost complete intersections do not have the same Hilbert function as an LPP almost complete intersection, and Migliore and Miró-Roig can, in a lot of cases, give the precise resolution of generic almost complete intersections.

The paper is organized in the following way. In Section 2, we outline our strategy using almost complete intersection monomial ideals and prove some preliminary lemmas. We then modify the attack used on monomial ideals so that it works in the nonmonomial case in Section 3. In this section, we also give an explicit comparison map between two Koszul complexes that is vital in detecting nonminimality in the mapping cone resolutions we use. In Section 4, we identify complete intersection ideals that can have the Hilbert function of an LPP almost complete intersection and examine their resolutions. We handle final possibilities in Section 5, where we also obtain some bounds on the Hilbert functions of almost complete intersections. The proof of the main result on graded Betti numbers is in Section 6.

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2 Resolutions of LPP almost complete intersections and the monomial case

Our first goal is to compare the resolutions of two almost complete intersection monomial ideals with the same Hilbert function. In this section, we assume that the ideals have minimal generators that form a maximal length regular sequence in the same degrees. Since we are restricting to almost complete intersections, we have some extra structure with which to work. We begin by describing the minimal resolution of a monomial almost complete intersection.

Let $L = (x_1^{a_1}, \ldots, x_n^{a_n}, x_1^{d_1} \cdots x_n^{d_n})$. We form the canonical short exact sequence

$$0 \to R/(x_1^{a_1-d_1}, \ldots, x_n^{a_n-d_n})(-\sum d_i) \to R/(x_1^{a_1}, \ldots, x_n^{a_n}) \to R/L \to 0.$$

We can find a resolution of R/L by taking the mapping cone induced by this short exact sequence. The situation here is particularly good because we have complete intersections in the first two nonzero places in the exact sequence, and therefore we can resolve these modules minimally by Koszul complexes.

The only difficulty is that this resolution of R/L might be nonminimal. Of course, the only places nonminimality may arise come from constants appearing in a comparison map between the resolutions of R/H and R/F, where $H = (x_1^{a_1-d_1}, \ldots, x_n^{a_n-d_n})$, and $F = (x_1^{a_1}, \ldots, x_n^{a_n})$. To see how to detect this nonminimality, we examine a comparison map closely in an example.

Let $L = (x_1^2, x_2^3, x_3^3, x_1x_2^2)$; this is a $(2,3,3)-$LPP ideal in $R = k[x_1, x_2, x_3]$. Let $F = (x_1^2, x_2^3, x_3^3)$, and let

$$H = (x_1^2, x_2^3, x_3^3) : (x_1x_2^2) = (x_1, x_2, x_3^3).$$

We resolve R/L using the following diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & R & \xrightarrow{\partial_3^F} & R^3 & \xrightarrow{\partial_2^F} & R^3 & \xrightarrow{\partial_1^F} & R & \longrightarrow & R/F & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 1} & & \uparrow{\scriptstyle C_2} & & \uparrow{\scriptstyle C_1} & & \uparrow{\scriptstyle x_1x_2^2} & & \uparrow{\scriptstyle x_1x_2^2} & & \\
0 & \longrightarrow & R & \xrightarrow[\partial_3^H]{} & R^3 & \xrightarrow[\partial_2^H]{} & R^3 & \xrightarrow[\partial_1^H]{} & R & \longrightarrow & R/H & \longrightarrow & 0
\end{array}$$

Here, the ∂_i^F and ∂_i^H are the Koszul maps. We need to determine what C_2 and C_1 are. It is not hard to see that C_2 should give the relationship between the generators of H and F, and from there, it is easy to compute the two maps. They are:

$$C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_2^2 & 0 \\ 0 & 0 & x_1 \end{pmatrix} \quad \text{and} \quad C_1 = \begin{pmatrix} x_2^2 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & x_1x_2^2 \end{pmatrix}.$$

Note that C_2 is just a diagonal matrix with diagonal entries the powers of the x_i that appear in the additional generator $x_1x_2^2$ of L. Moreover, C_1 is just the matrix of 2×2 minors of C_2, and $x_1x_2^2$ is the determinant of C_2. This suggests a general strategy: Compute the penultimate vertical map C in the diagram, and fill in the other vertical maps with the appropriate exterior powers of C. We shall show that these are the maps we want.

Lemma 2.1 *Let $L = (x_1^{a_1}, \ldots, x_n^{a_n}, x_1^{d_1} \cdots x_n^{d_n})$. Let $F = (x_1^{a_1}, \ldots, x_n^{a_n})$ and $H = (x_1^{a_1-d_1}, \ldots, x_n^{a_n-d_n})$, and let ∂_i^F and ∂_i^H be the Koszul maps. Let $C : \bigwedge^{n-1} R^n \to \bigwedge^{n-1} R^n$ be given by $e_A \mapsto x_i^{d_i} e_A$, where $A = \{1, 2, \ldots, \hat{\imath}, \ldots, n\}$. Then the following diagram is commutative for all $j = 1, \ldots, n$:*

$$\begin{array}{ccc}
\bigwedge^j R^n & \xrightarrow{\partial_j^F} & \bigwedge^{j-1} R^n \\
{\scriptstyle \bigwedge^{n-j} C}\uparrow & & \uparrow{\scriptstyle \bigwedge^{n-j+1} C} \\
\bigwedge^j R^n & \xrightarrow[\partial_j^H]{} & \bigwedge^{j-1} R^n
\end{array}$$

Proof: Let $A = \{i_1, \ldots, i_j\} \subset [n] = \{1, \ldots, n\}$, and let $e_A = e_{i_1} \wedge \cdots \wedge e_{i_j}$ be a basis element of $\bigwedge^j R^n$ in the resolution of R/H. Note that C is just the diagonal matrix with diagonal entries $x_i^{d_i}$ such that $\partial_n^F \circ 1 = C \circ \partial_n^H$. We have

$$\partial_j^H(e_A) = \sum_{i_t \in A} (-1)^t x_{i_t}^{a_{i_t} - d_{i_t}} e_{A \setminus i_t}.$$

Applying $\bigwedge^{n-j+1} C$, we obtain

$$\left(\bigwedge^{n-j+1} C \circ \partial_j^H\right)(e_A) = \sum_{i_t \in A} (-1)^t x_{i_t}^{a_{i_t} - d_{i_t}} \left(\prod_{l \notin A \setminus i_t} x_l^{d_l}\right) e_{A \setminus i_t}.$$

$$= \sum_{i_t \in A} (-1)^t x_{i_t}^{a_{i_t}} \left(\prod_{l \notin A} x_l^{d_l}\right) e_{A \setminus i_t}.$$

Now, going the other direction,

$$\left(\bigwedge^{n-j} C\right)(e_A) = \left(\prod_{l \notin A} x_l^{d_l}\right) e_A,$$

and then

$$\left(\partial_j^F \circ \bigwedge^{n-j} C\right)(e_A) = \sum_{i_t \in A} (-1)^t x_{i_t}^{a_{i_t}} \left(\prod_{l \notin A} x_l^{d_l}\right) e_{A \setminus i_t}.$$

The coefficients of each $e_{A \setminus i_t}$ are equal, and thus we are done. □

As a result, we can easily determine the nonminimality that occurs in the mapping cone resolution of an almost complete intersection monomial ideal. The matrix C in Lemma 2.1 has entry 1 only in the places corresponding to some d_i being 0, forcing cancellation in the tail of the resolution in degree $a_1 + \cdots + \hat{a}_i + \cdots + a_n$. Since the other vertical maps are matrices of minors of C, we can detect constants in the other vertical maps just by knowing C.

We use Lemma 2.1 to compare the resolutions of two monomial almost complete intersections. We show that if there is any nonminimality in the mapping cone resolution of an LPP almost complete intersection, there is corresponding nonminimality in the mapping cone resolution of the other ideal we are considering.

Proposition 2.2 *Let I be a monomial ideal containing minimal generators $x_1^{a_1}, \ldots, x_n^{a_n}$. Suppose I has the same Hilbert function as an $(a_1, \ldots, a_n)$−LPP almost complete intersection L. Then $\beta^{R/I} \leq \beta^{R/L}$.*

Proof: We may immediately reduce to the case in which we can apply Lemma 2.1: The theorem of Clements and Lindström in [4] shows that L has at least as many generators in each degree as I, and therefore I must also be an almost complete intersection. (Alternatively, see Lemma 5.1.) Hence we may assume that $I = (x_1^{a_1}, \ldots, x_n^{a_n},\ x_1^{b_1} \cdots x_n^{b_n})$. Because I and L have the same Hilbert function and have as minimal generators the same powers of the x_i, the sets $S_d = \{a_1 - d_1, \ldots, a_n - d_n\}$ and $S_b = \{a_1 - b_1, \ldots, a_n - b_n\}$ must be the same (for we can compute the Hilbert functions from the nonminimal mapping cone resolutions). We show that if some d_i is zero, then there exists an i' (a different one for each i with $d_i = 0$) such that $a_{i'} = a_i$ and $b_{i'} = 0$. Since any constant in the comparison map (aside from the far left vertical map, which always induces nonminimality) comes from some b_i or d_i being zero, this proves that any non-minimality in the mapping cone resolution of R/L occurs in the same degree in the resolution of R/I.

It is clear from the definition of an LPP ideal that $d_1 = a_1 - 1, \ldots, d_{j-1} = a_{j-1} - 1$, and $d_{j+1} = \cdots = d_n = 0$ for some $1 < j \leq n$. Hence $S_d = \{1, \ldots, 1, a_j - d_j,\ a_{j+1}, \ldots, a_n\}$ for some $1 < j \leq n$, and since the a_i are weakly increasing, $1 \leq a_j - d_j \leq a_{j+1} \leq \cdots \leq a_n$.

If no $d_i = 0$, then the mapping cone resolution of R/L is minimal (except for the obvious nonminimality from the far left vertical map), and we are done. Otherwise, $d_n = 0$. Since $S_b = S_d$, $a_n \in S_b$, and there is some r such that $a_n = a_r - b_r$. Because $a_n \geq a_i$ for all i, we must have $a_r = a_n$ and $b_r = 0$.

Inductively, suppose that for $s > l$, the following holds: If $d_s = 0$, there is s' such that $a_{s'} = a_s$ and $b_{s'} = 0$. If $d_l > 0$, there is nothing to prove. Otherwise, $d_l = 0$, and then $a_l \in S_d = S_b$. By the induction hypothesis, the only way for a_l to be in S_b is for there to be an $a_{l'} = a_l$ such that $b_{l'} = 0$ (for any a_i larger than a_l has $b_i = 0$ by induction), with l' different from the other indices we have already used. This is what we needed to prove, and therefore we are done by Lemma 2.1. $\square$

This establishes the inequality we want for the monomial case and suggests a method of attack for the nonmonomial case, which we explore in the next section.

3 The nonmonomial almost complete intersection case

It was particularly convenient to work with monomial ideals in the last section because the comparison map consists of diagonal matrices of a nice form. Unfortunately, if one wishes to generalize Proposition 2.2 to nonmonomial ideals, the vertical maps will be much more complicated. We mimic the proof for monomial ideals in this section.

Throughout, fix an LPP ideal $L = (x_1^{a_1}, \ldots, x_n^{a_n}, x_1^{d_1} \cdots x_n^{d_n})$. Let $I = (f_1, \ldots, f_n, g)$ be an ideal with the same Hilbert function as L such that $\deg f_i =$

a_i, the f_i form a regular sequence, and $(f_1, \ldots, f_n) : (g) = (h_1, \ldots, h_n)$ is a complete intersection. The reason for the last hypothesis is that we want to follow the monomial case; we get a resolution of R/I by taking the mapping cone of two Koszul complexes.

The goal in this section is Theorem 3.3, the LPP Conjecture in the setting outlined above. To compare the graded Betti numbers of R/I and R/L, we first find a convenient comparison map between the Koszul complexes on the f_i and h_j. Let $F = (f_1, \ldots, f_n)$ and $H = (h_1, \ldots, h_n)$. The strategy is to find a matrix C that writes the f_i in terms of the h_j and then to fill in the other vertical maps with the exterior powers of C. This gives us a comparison map in which we can detect the presence of nonzero constants in any of the vertical maps solely by examining C.

Lemma 3.1 *Let ∂_i^F and ∂_i^H be the Koszul maps in the resolutions of R/F and R/H. Let C be any lift induced by g from the Comparison Theorem mapping $\bigwedge^{n-1} R^n$ in the resolution of R/H to $\bigwedge^{n-1} R^n$ in the resolution of R/F. Then the following diagram is commutative for all $j = 1, \ldots, n$:*

$$\begin{array}{ccc} \bigwedge^j R^n & \xrightarrow{\partial_j^F} & \bigwedge^{j-1} R^n \\ {\scriptstyle \bigwedge^{n-j} C}\uparrow & & \uparrow{\scriptstyle \bigwedge^{n-j+1} C} \\ \bigwedge^j R^n & \xrightarrow[\partial_j^H]{} & \bigwedge^{j-1} R^n \end{array}$$

Proof: Let $A = \{i_1, \ldots, i_j\} \subset [n]$, and let $e_A = e_{i_1} \wedge \cdots \wedge e_{i_j}$ be a basis element of $\bigwedge^j R^n$ in the resolution of R/H. Throughout, for $X, Y \subset [n]$, let $m_{(X,Y)}$ represent the minor of C obtained by omitting rows X and columns Y from C.

Computing first $(\bigwedge^{n-j+1} C \circ \partial_j^H)(e_A)$, we have

$$\partial_j^H(e_A) = \sum_{l=1}^{j} (-1)^l h_{i_l} e_{A \setminus i_l},$$

and applying $\bigwedge^{n-j+1} C$, we obtain

$$\sum_{l=1}^{j} (-1)^l h_{i_l} \sum_{|B|=j-1} m_{(B, A\setminus i_l)} e_B = \sum_{|B|=j-1} \sum_{l=1}^{j} (-1)^l h_{i_l} m_{(B, A\setminus i_l)} e_B,$$

where the summation is over all $B \subset [n]$ of cardinality $j - 1$.

In the other direction,

$$\left(\bigwedge^{n-j} C \right)(e_A) = \sum_{|D|=j} m_{(D,A)} e_D,$$

with the summation over all $D \subset [n]$ of cardinality j. Applying ∂_j^F yields

$$\sum_{|D|=j} m_{(D,A)} \sum_{l=1}^{j} (-1)^l f_{D_l} e_{D \setminus D_l},$$

where D_l represents the l^{th} element of D in increasing order.

To compare these two calculations, we convert the second to a summation over sets of cardinality $j-1$. The messy part here is keeping track of the signs. Let $p(x, Y)$ denote the position of x in the set $Y \subset [n]$, where Y is ordered in the usual way. Then

$$\left(\partial_j^F \circ \bigwedge^{n-j} C\right)(e_A) = \sum_{|B|=j-1} \sum_{\alpha \notin B} (-1)^{p(\alpha, B\cup\alpha)} f_\alpha m_{(B\cup\alpha, A)} e_B.$$

We need to write the f_i in terms of the h_j. We index the entries of the matrix C unconventionally to keep the notation simpler. We have

$$\begin{pmatrix} c_{nn} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{11} \end{pmatrix} \begin{pmatrix} (-1)^n h_n \\ \vdots \\ -h_1 \end{pmatrix} = \begin{pmatrix} (-1)^n f_n \\ \vdots \\ -f_1 \end{pmatrix}.$$

Thus $(-1)^i f_i = (-1)^n c_{in} h_n + \cdots + (-1)^1 c_{i1} h_1$. Hence

$$\left(\partial_j^F \circ \bigwedge^{n-j} C\right)(e_A) = \sum_{|B|=j-1} \sum_{\alpha \notin B} (-1)^{p(\alpha, B\cup\alpha)} m_{(B\cup\alpha, A)} (-1)^\alpha \sum_{r=1}^{n} (-1)^r c_{\alpha r} h_r e_B.$$

Suppose first that $r \in A$, so $r = i_l$ for some l. We show that the coefficients of $h_{i_l} e_B$ in the two computations are identical by proving that

$$(-1)^l m_{(B, A\setminus i_l)} = \sum_{\alpha \notin B} (-1)^{p(\alpha, B\cup\alpha)} m_{(B\cup\alpha, A)} (-1)^\alpha (-1)^{i_l} c_{\alpha i_l}.$$

Because

$$m_{(B, A\setminus i_l)} = \sum_{\alpha \notin B} (-1)^{p(\alpha, B')+p(i_l, A'\cup i_l)} c_{\alpha i_l} m_{(B\cup\alpha, A)},$$

where X' is the complement of X in $[n]$, it is enough to show that

$$l + p(\alpha, B') + p(i_l, A'\cup i_l) \equiv p(\alpha, B\cup\alpha) + \alpha + i_l \pmod 2,$$

which would prove that the signs are consistent. Note that $l = p(i_l, A)$, and for $x \in Y$, $p(x, Y) + p(x, Y'\cup x) = x + 1$, so the left-hand side is just

$$p(\alpha, B') + i_l + 1.$$

The right-hand side is

$$\alpha + 1 - p(\alpha, B') + \alpha + i_l,$$

and thus the parities are equal.

Finally, if instead $r \notin A$, we need the coefficient of $h_r e_B$ in the expression for $(\partial_j^F \circ \bigwedge^{n-j} C)(e_A)$ to be zero for all B. This follows because this formula includes the Laplacian determinant expansion of a submatrix of C with column r repeated. □

To prove that the $\bigwedge^j C$ form a comparison map, we still need to show that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi_F} & R/F \\ {\scriptstyle \det C}\uparrow & & \uparrow{\scriptstyle g} \\ R & \xrightarrow[\pi_H]{} & R/H \end{array}$$

is commutative, where π_F and π_H are the canonical projections. This follows from the next lemma.

Lemma 3.2 *Let C be as above. Then $\det C = g$ in R/F.*

Proof: We suppose that $n \geq 3$ since when $n = 2$, our goal, Theorem 3.3, is trivial by the Hilbert-Burch Theorem in the graded case (see, e.g., [3]). We shall use a diagram chase to show that $\det C - g \in F$, which gives the result.

For $j = 2, \ldots, n-1$, let A_j be vertical maps induced by g from $\bigwedge^{n-j} R^n$ in the resolution of R/H to $\bigwedge^{n-j} R^n$ in the resolution of R/F. Also, throughout, let M_i be matrices of appropriate size with their entries homogeneous polynomials. The following diagram is commutative:

$$\begin{array}{ccccccccc} \bigwedge^{n-1} R^n & \xrightarrow{\partial^F_{n-1}} & \bigwedge^{n-2} R^n & \xrightarrow{\partial^F_{n-2}} & \cdots & \xrightarrow{\partial^F_2} & \bigwedge^1 R^n & \xrightarrow{\partial^F_1} & \bigwedge^0 R^n \\ {\scriptstyle C-C}\uparrow & & {\scriptstyle \bigwedge^2 C - A_2}\uparrow & & & & \uparrow{\scriptstyle \bigwedge^{n-1} C - A_{n-1}} & & \uparrow{\scriptstyle \det C - g} \\ \bigwedge^{n-1} R^n & \xrightarrow[\partial^H_{n-1}]{} & \bigwedge^{n-2} R^n & \xrightarrow[\partial^H_{n-2}]{} & \cdots & \xrightarrow[\partial^H_2]{} & \bigwedge^1 R^n & \xrightarrow[\partial^H_1]{} & \bigwedge^0 R^n \end{array}$$

Because $\left(\bigwedge^2 C - A_2\right) \circ \partial^H_{n-1}$ is zero,

$$\operatorname{im}\left(\bigwedge^2 C - A_2\right)^T \subseteq \ker(\partial^H_{n-1})^T = \operatorname{im}(\partial^H_{n-2})^T.$$

Hence $(\bigwedge^2 C - A_2) = M_2 \circ \partial^H_{n-2}$ for some M_2. Commutativity of the next square to the right yields

$$\left(\bigwedge^3 C - A_3\right) \circ \partial^H_{n-2} = \partial^F_{n-2} \circ \left(\bigwedge^2 C - A_2\right) = \partial^F_{n-2} \circ M_2 \circ \partial^H_{n-2}.$$

Therefore,

$$\operatorname{im}\left(\left(\bigwedge^3 C - A_3\right) - \partial^F_{n-2} \circ M_2\right)^T \subseteq \ker(\partial^H_{n-2})^T = \operatorname{im}(\partial^H_{n-3})^T.$$

Thus for some M_3, we have

$$\left(\bigwedge^3 C - A_3\right) = \partial^F_{n-2} \circ M_2 + M_3 \circ \partial^H_{n-3}.$$

Inductively, suppose that there exist M_{j-2} and M_{j-1} such that

$$\bigwedge^{j-1} C - A_{j-1} = \partial^F_{n-j+2} \circ M_{j-2} + M_{j-1} \circ \partial^H_{n-j+1}.$$

By commutativity,

$$\left(\bigwedge^j C - A_j\right) \circ \partial^H_{n-j+1} = \partial^F_{n-j+1} \circ \left(\bigwedge^{j-1} C - A_{j-1}\right),$$

so

$$\left(\bigwedge^j C - A_j\right) \circ \partial^H_{n-j+1} = \partial^F_{n-j+1} \circ (\partial^F_{n-j+2} \circ M_{j-2} + M_{j-1} \circ \partial^H_{n-j+1}).$$

Thus, as in the arguments above,

$$\left(\bigwedge^j C - A_j\right) - \partial^F_{n-j+1} \circ M_{j-1} = M_j \circ \partial^H_{n-j}$$

for some M_j. Consequently,

$$\bigwedge^{n-1} C - A_{n-1} = M_{n-1} \circ \partial^H_1 + \partial^F_2 \circ M_{n-2}$$

for some M_{n-2} and M_{n-1}. Commutativity of the rightmost square in the diagram gives

$$(\det C - g) \circ \partial^H_1 = \partial^F_1 \circ \left(\bigwedge^{n-1} C - A_{n-1}\right),$$

and so

$$(\det C - g) \circ \partial^H_1 = \partial^F_1 \circ M_{n-1} \circ \partial^H_1.$$

As a result,

$$(\det C - g) = (f_1 \cdots f_n)(p_1 \cdots p_n)^T$$

for some homogeneous polynomials p_i, and therefore $\det C - g \in F$. □

We now have a convenient, explicit comparison map, and therefore we can examine the mapping cone resolution of R/I and detect the nonminimality.

Theorem 3.3 *Let $L = (x_1^{a_1}, \ldots, x_n^{a_n}, x_1^{d_1} \cdots x_n^{d_n})$ be an LPP ideal. Let $I = (f_1, \ldots, f_n, g)$ be an ideal with the same Hilbert function as L such that* $\deg f_i = a_i$*, the f_i form a regular sequence, and $(f_1, \ldots, f_n) : (g) = (h_1, \ldots, h_n)$ is a complete intersection. Then $\beta^{R/I} \leq \beta^{R/L}$.*

Proof: We need to determine where there are nonzero constants in the comparison map between the resolution of R/H and that of R/F. We show that there are nonzero constants in the columns of the vertical maps, each in a different row, that correspond to the degrees in which there is nonminimality in the LPP mapping cone resolution.

Let C be the map used in the mapping cone resolution of R/I as above. Index the h_i such that $\deg h_i = a_i - d_i$; then $\deg h_1 \leq \cdots \leq \deg h_n$. Suppressing signs, by commutativity, we have

$$\begin{pmatrix} c_{nn} & \cdots & c_{n1} \\ \vdots & \ddots & \vdots \\ c_{1n} & \cdots & c_{11} \end{pmatrix} \begin{pmatrix} h_n \\ \vdots \\ h_1 \end{pmatrix} = \begin{pmatrix} f_n \\ \vdots \\ f_1 \end{pmatrix},$$

with the indexing of the entries C done unconventionally as in Lemma 3.1.

Let us consider the entries of C. Up to signs,

$$c_{in}h_n + \cdots + c_{ii}h_i + \cdots c_{i1}h_1 = f_i.$$

We have $\deg f_i \geq \deg h_i$ for all i. Note that $\deg f_i = \deg h_i$ if and only if $d_i = 0$. In that case, c_{ii} is a constant, and since $\deg h_i = \deg f_i \geq \cdots \geq \deg f_1$, all of $c_{ii}, \ldots, c_{1i}$ must be constants. Thus a typical column of C has entries with positive degree at the top, possibly constant entries in the middle, and zeros at the bottom when $\deg h_i > \deg f_j$ for j small.

Suppose that c_{ii} and c_{ji} are both nonzero constants in column i of C. Then $d_i = 0$, and $\deg f_i = \deg h_i = \deg f_j$. We may perform row operations on C to change one of these nonzero constants to zero, modifying the generating set of F to be the appropriate linear combination of f_i and f_j. In this way, we may assume that each column of C contains at most one nonzero constant. Moreover, by reindexing generators of F with the same degree, we may suppose that if $d_i = 0$ and there is a nonzero constant in column i, it occurs on the diagonal in position c_{ii}. We assume C has this form in the rest of the proof. (Note that there may also be nonzero constants below the main diagonal in a column j of C when $d_j \neq 0$; see Example 1 below.)

If all $d_i > 0$, then there is no nonminimality in the LPP mapping cone resolution (apart from that arising from the map $\bigwedge^n R^n \to \bigwedge^n R^n$), so we are done. If not, then $d_n = 0$, and so $\deg h_n = \deg f_n$. Therefore c_{nn} is a constant. Since $\deg h_n = \deg f_n \geq \deg f_j$ for all j, c_{jn} is a constant for all j. Suppose all $c_{jn} = 0$. Then we can write all the f_i in terms of only $h_1, \ldots, h_{n-1}$. Hence $(f_1, \ldots, f_n) \subset (h_1, \ldots, h_{n-1})$, meaning that a depth n ideal is contained in a depth $n-1$ ideal, which is impossible. Thus some c_{jn} is a nonzero constant, and it must be c_{nn} by our assumptions on the form of C.

Our objective is to show that for each d_i that is zero, there is a nonzero constant in column i of the matrix C and that these occur in different rows. This proves that if there is nonminimality in the mapping cone resolution of R/L, nonminimality occurs in the same degree in the same place in the mapping cone resolution of R/I since the comparison map is made up of the exterior powers of C.

We proceed by induction. The base case is above, showing that if $d_n = 0$, then c_{nn} is a nonzero constant. For the induction hypothesis, suppose that for each $j > v$ such that $d_j = 0$, c_{jj} is a nonzero constant. If $d_v \neq 0$, there is nothing to prove. Suppose $d_v = 0$. Then $\deg h_v = a_v - 0 = \deg f_v$. Since, up to signs,

$$f_v = c_{vn}h_n + \cdots + c_{vv}h_v + \cdots + c_{v1}h_1,$$

c_{vv} must be a constant. As before, $\deg h_v = \deg f_v \geq \deg f_i$ for all $i \leq v$, and hence c_{iv} is a constant for all $i \leq v$.

We show that c_{vv} is a nonzero constant. Suppose that for all $i \leq v$, $c_{iv} = 0$. Because $\deg h_{v+r} \geq \deg h_v = \deg f_v$ for all $r \geq 0$, $c_{vn}, \ldots, c_{vv}$ are all constants. Moreover, $\deg h_v = \deg f_v \geq \deg f_{v-s}$ for all $s \geq 0$. These inequalities mean that c_{ij} is a constant when both $i \leq v$ and $j \geq v$. That is, every bolded entry in the lower left-hand corner of C shown below

$$\begin{pmatrix} c_{nn} & \cdots & c_{n,v+1} & c_{nv} & c_{n,v-1} & \cdots & c_{n1} \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ c_{v+1,n} & \cdots & c_{v+1,v+1} & c_{v+1,v} & c_{v+1,v-1} & \cdots & c_{v+1,1} \\ \mathbf{c_{vn}} & \cdots & \mathbf{c_{v,v+1}} & \mathbf{c_{vv}} & c_{v,v-1} & \cdots & c_{v1} \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ \mathbf{c_{1n}} & \cdots & \mathbf{c_{1,v+1}} & \mathbf{c_{1v}} & c_{1,v-1} & \cdots & c_{11} \end{pmatrix}$$

is a constant. We are assuming that $c_{vv}, \ldots, c_{1v}$ are all zero, and by our assumptions on the form of C, we conclude that all the entries in the bolded corner are zero. That means that we can write $f_v, \ldots, f_1$ in terms of only $h_{v-1}, \ldots, h_1$, and thus $(f_1, \ldots, f_v) \subset (h_1, \ldots, h_{v-1})$. This implies that an ideal of depth v is contained in an ideal of depth $v - 1$, which is a contradiction. Therefore some c_{iv} must be a nonzero constant, and it must be c_{vv} by our construction of C.

With that established, the only question is whether having the correct nonminimality from the matrix C gives us nonzero constants in the right columns of the other vertical maps. The argument above shows that C has leading nonzero constants in the columns for which $d_i = 0$, each in a different row. Therefore the minors of C have nonzero constants in the columns corresponding to subsets of the columns in which C has nonzero constants, which is what we need. Hence any nonminimality that occurs in the mapping cone resolution of R/L occurs in the same degree in that of R/I, and thus $\beta^{R/I} \leq \beta^{R/L}$. $\square$

We give two examples to illustrate some of the ideas from the previous proof. The first uses monomial ideals for simplicity and demonstrates how the graded Betti numbers of the LPP ideal can be strictly larger than those of the other ideal. The second focuses on the nonmonomial case and also shows the form we assume the matrix C to have in the proof above.

Example 1: Let $R = \mathbb{C}[a, b, c, d]$, $L = (a^2, b^3, c^4, d^5, ab^2c^2)$, and $I = (a^2, b^3, c^4, d^5, b^2c^3)$. Then L is a $(2, 3, 4, 5)-$LPP ideal, and L and I have the same

Hilbert function. Note that

$$(a^2, b^3, c^4, d^5) : (ab^2c^2) = (a, b, c^2, d^5) \text{ and } (a^2, b^3, c^4, d^5) : (b^2c^3) = (a^2, b, c, d^5).$$

We write a^2, b^3, c^4, d^5 in terms of the generators of the ideal quotients, ordering everything as in the proof of Theorem 3.3, to give an example of the procedure followed there. The expression for L is on the left, and the one for I is on the right. We have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c^2 & 0 & 0 \\ 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} d^5 \\ c^2 \\ b \\ a \end{pmatrix} = \begin{pmatrix} d^5 \\ c^4 \\ b^3 \\ a^2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c^3 \\ 0 & 0 & b^2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} d^5 \\ a^2 \\ b \\ c \end{pmatrix} = \begin{pmatrix} d^5 \\ c^4 \\ b^3 \\ a^2 \end{pmatrix}.$$

There is a one in the upper left-hand corner of both matrices, which corresponds to the power of d being zero in ab^2c^2, the extra generator of L. Thus the same nonminimality from this nonzero constant will occur in both mapping cone resolutions. There is also a one in the matrix for I in the bottom row, and therefore there will be additional nonminimality in the mapping cone resolution of R/I. Hence we expect the ranks of the last three free modules in the minimal resolution of R/L to be respectively one, two, and one greater than the ranks in the minimal resolution of R/I.

The minimal resolution of R/L is

$$0 \to R^3 \to R^9 \to R^{10} \to R^5 \to R \to R/L \to 0,$$

while the minimal resolution of R/I is

$$0 \to R^2 \to R^7 \to R^9 \to R^5 \to R \to R/I \to 0.$$

There are two copies of $R(-12)$ and $R(-7)$ in the minimal resolution of R/L that do not appear in that of R/I, and otherwise the graded Betti numbers are the same.

Example 2: For a simple example involving a nonmonomial ideal, we consider the case of five quadrics in four variables. Let $R = \mathbb{C}[a, b, c, d]$, and let $f_1 = ad + d^2$, $f_2 = c^2 - bd$, $f_3 = ab - b^2$, $f_4 = a^2 - bc$, and $g = a^2 + cd$. Note that the f_i form a regular sequence and that $(f_1, f_2, f_3, f_4) : g$ is a complete intersection. Let $I = (f_1, f_2, f_3, f_4, g)$ and $L = (a^2, b^2, c^2, d^2, ab)$. Then L is a $(2, 2, 2, 2)-$LPP ideal, and $H_{R/I} = H_{R/L}$. Theorem 3.3 says that the graded Betti numbers of R/I are at most those of R/L; we verify this.

In the extra generator ab of L, the powers of c and d are both zero, so we should have nonminimality in the mapping cone resolution of R/L. Let $J = (a^2, b^2, c^2, d^2)$, and let $M = J : (ab) = (a, b, c^2, d^2)$. Suppressing gradings for the sake of room, we have

$$\begin{array}{ccccccccccc} R & \xrightarrow{\partial_4^J} & R^4 & \xrightarrow{\partial_3^J} & R^6 & \xrightarrow{\partial_2^J} & R^4 & \xrightarrow{\partial_1^J} & R & \longrightarrow & R/J \\ {\scriptstyle 1}\big\uparrow & & {\scriptstyle C^L}\big\uparrow & & {\scriptstyle C_2^L}\big\uparrow & & \big\uparrow{\scriptstyle C_1^L} & & \big\uparrow{\scriptstyle ab} & & \big\uparrow{\scriptstyle ab} \\ R & \xrightarrow[\partial_4^M]{} & R^4 & \xrightarrow[\partial_3^M]{} & R^6 & \xrightarrow[\partial_2^M]{} & R^4 & \xrightarrow[\partial_1^M]{} & R & \longrightarrow & R/M \end{array}$$

with C_2^L and C_1^L the exterior powers of C^L. Clearly the far left map induces cancellation. Since C^L is a diagonal matrix with nonzero entries $1, 1, a$, and b, there are two columns with nonzero constants in C^L, and there is one in C_2^L. Thus the minimal resolution of R/L is

$$0 \to R^2 \to R^7 \to R^9 \to R^5 \to R \to R/L \to 0.$$

Now, we have

$$I : (g) = (h_1, h_2, h_3, h_4) = (c, a - b + d, bd, b^2 + d^2),$$

and we can write the f_i in terms of the h_j as

$$\begin{pmatrix} 1 & -2 & a+b-d & -b \\ 0 & -1 & b & 0 \\ 0 & -1 & 0 & c \\ 0 & 1 & d & 0 \end{pmatrix} \begin{pmatrix} h_4 \\ h_3 \\ h_2 \\ h_1 \end{pmatrix} = \begin{pmatrix} f_4 \\ f_3 \\ f_2 \\ f_1 \end{pmatrix}.$$

We can convert the above 4×4 matrix into a matrix C^I that has at most one nonzero constant in each column. We accomplish this by changing the generating set of F to $(f_4 + 2f_1, f_1, f_1 + f_2, f_1 + f_3)$, and we get

$$\begin{pmatrix} 1 & 0 & a+b+d & -b \\ 0 & 1 & d & 0 \\ 0 & 0 & d & c \\ 0 & 0 & b+d & 0 \end{pmatrix} \begin{pmatrix} h_4 \\ h_3 \\ h_2 \\ h_1 \end{pmatrix} = \begin{pmatrix} f_4 + 2f_1 \\ f_1 \\ f_1 + f_2 \\ f_1 + f_3 \end{pmatrix}.$$

Thus C^I has nonzero constants in two columns, in separate rows, and induces the same number of cancellations as in the mapping cone resolution of R/L. The cancellations occur in the same degrees, and the minimal resolutions of R/I and R/L are both

$$0 \to R(-7)^2 \to R(-5)^4 \oplus R(-6)^3 \to R(-3)^2 \oplus R(-4)^7 \to R(-2)^5 \to R \to 0.$$

In the next sections, we remove some of the restrictions we have placed on I in Section 3.

4 Complete intersections

So far we have compared the graded Betti numbers of an LPP almost complete intersection L to another almost complete intersection I. In this section, we explore what happens when I is instead a complete intersection. As a consequence, we also obtain a better idea of what ideals with regular sequence in degrees $a_1, \dots, a_n$ can have the same Hilbert function as an $(a_1, \dots, a_n)-$LPP almost complete intersection.

Our goal is the following result:

Proposition 4.1 *Let L be an $(a_1, \ldots, a_n)-$LPP almost complete intersection. Let I be any complete intersection ideal with the same Hilbert function as L that contains a regular sequence in degrees $a_1, \ldots, a_n$. Then $\beta^{R/I} \leq \beta^{R/L}$.*

To prove this, we first determine what form L and I must have to satisfy the hypotheses of Proposition 4.1. We may assume that I is a monomial ideal since its resolution and Hilbert function depend only on the degrees of the generators. Of course, for I to contain a regular sequence in degrees $a_1, \ldots, a_n$, the minimal generators of I must be in degrees at most $a_1, \ldots, a_n$. Our first step is to rule out all but one type of candidate for L.

Lemma 4.2 *If L and I are as above, then $L = (x_1^{a_1}, \ldots, x_n^{a_n}, x_1^{a_1-1}x_2^{b_2})$, where $1 \leq b_2 \leq a_2 - 2$.*

Proof: R/L and R/I have the same Hilbert function and are Artinian, and hence they have the same regularity and Betti number 1 in the same highest degree in the n^{th} term of their minimal resolutions. Suppose $L = (x_1^{a_1}, \ldots, x_n^{a_n}, x_1^{a_1-1} \cdots x_j^{a_j-1} x_{j+1}^{b_{j+1}})$. We get a short exact sequence

$$0 \to R/(x_1, \ldots, x_j, x_{j+1}^{a_{j+1}-b_{j+1}}, x_{j+1}^{a_{j+2}}, \ldots, x_n^{a_n}) \to R/(x_1^{a_1}, \ldots, x_n^{a_n}) \to R/L \to 0,$$

with the first term shifted in degree by $-(a_1 + \cdots + a_j + b_{j+1} - j)$. This induces a mapping cone resolution of R/L. The degrees of the generators of the n^{th} term F_n of the minimal resolution of R/L come from sums of $n-1$ choices of $1, \ldots, 1$, $a_{j+1} - b_{j+1}, a_{j+2}, \ldots, a_n$ plus the shift in degree of $(a_1 - 1) + \cdots (a_j - 1) + b_{j+1}$, but the only combinations available are those that are not canceled because of nonminimality. Thus we must leave out either a 1 or $a_{j+1} - b_{j+1}$, or else that portion of the chain map is $1 = x_r^0$ for some $r > j+1$. To get the largest degree possible, we leave out a 1, giving us the degree of the highest degree generator of F_n. But we may have only one of these. Hence $j = 1$, and $a_2 - b_2 \geq 2$, so $1 \leq b_2 \leq a_2 - 2$. □

Lemma 4.2 tells us what L must look like under the hypotheses of Proposition 4.1. The next two lemmas place strong restrictions on I.

Lemma 4.3 *With L and I as above, $I = (x_1^{a_1}, \ldots, x_i^{a_i-1}, \ldots, x_n^{a_n})$, up to reindexing.*

Proof: Write $I = (x_1^{d_1}, \ldots, x_n^{d_n})$. Since I contains a regular sequence in degrees $a_1, \ldots, a_n$, reindexing if necessary, $d_j \leq a_j$ for all j. Let $d_j = a_j - c_j$. Then the degree of the generator for the n^{th} term in the resolution of R/I is the sum of the $a_j - c_j$. Note that

$$(x_1^{a_1}, \ldots, x_n^{a_n}) : (x_1^{a_1-1}x_2^{b_2}) = (x_1, x_2^{a_2-b_2}, x_3^{a_3}, \ldots, x_n^{a_n}).$$

Thus the highest degree generator in the n^{th} term of the resolution of R/L has degree $(a_1 - 1 + b_2) + (a_2 - b_2) + a_3 + \cdots + a_n$, which comes from leaving out 1 in the set of degrees of minimal generators in the ideal quotient. Therefore

$$\sum_{j=1}^{n}(a_j - c_j) = (a_1 - 1 + b_2) + (a_2 - b_2) + \sum_{j=3}^{n} a_j = \sum_{j=1}^{n} a_j - 1.$$

Hence some $c_j = 1$, and the rest are 0. □

Lemma 4.4 *For L and I as above to have the same Hilbert function, either $i = 2$, or $a_i = a_2$.*

Proof: Suppose the conclusion is false. We show that the Hilbert functions are then different, and we proceed by proving that the alternating sum of Betti numbers for a certain degree is nonzero for R/L and zero for R/I. We know that $I = (x_1^{a_1}, \ldots, x_i^{a_i-1}, \ldots, x_n^{a_n})$, where the a_i are weakly increasing, and $L = (x_1^{a_1}, \ldots, x_n^{a_n}, x_1^{a_1-1}x_2^{a_2-l})$, where $l \geq 2$. Suppose that $i > 2$ and $a_i > a_2$. The degrees of the elements in the last term of the resolution of R/L are

$$1 + \sum_{j=3}^{n} a_j + (a_1 - 1) + (a_2 - l) = \sum_{j=1}^{n} a_j - l \quad \text{and}$$

$$l + \sum_{j=3}^{n} a_j + (a_1 - 1) + (a_2 - l) = \sum_{j=1}^{n} a_j - 1.$$

We claim that $\beta_{md}^{R/L} = 0$ for $m < n$ and $d = a_1 + \cdots + a_n - l$, and $\beta_{md}^{R/I} = 0$ for all m. This would prove the lemma since $\beta_{nd}^{R/L} = 1$.

Consider first R/I. We need to prove that no subset of $\{a_1, \ldots, a_i - 1, \ldots, a_n\}$ sums to give d. This is equivalent to showing that no subset sums to $l-1$ because $a_1 + \cdots + a_i - 1 + \cdots a_n = d + l - 1$. If such a subset exists, it must consist of a_1 alone since $a_j > l$ for $j > 1$. Note that $a_i - 1 = (a_1 - 1) + (a_2 - l)$ since the degree of $x_i^{a_i-1}$ must equal the degree of the extra generator of L, so $a_i = a_1 + a_2 - l$. If $a_1 = l - 1$, then $a_i = a_2 - 1$, and we have a contradiction because $a_i \geq a_2$. Thus $\beta_{md}^{R/I} = 0$ for all m.

Next consider the resolution of R/L. Recall that we obtain generators of the free modules in the resolution of a certain degree in one of two ways: Either we take sums of a subset of $\{a_1, \ldots, a_n\}$, or we take sums of a subset of $\{1, l, a_3, \ldots, a_n\}$ and add $(a_1 - 1) + (a_2 - l)$. Suppose that some subset of $\{a_1, \ldots, a_n\}$ sums to d. Then the complementary subset sums to l, and because $a_j > l$ for $j > 1$, $a_1 = l$. But then $a_i - 1 = (a_1 - 1) + (a_2 - l)$, so $a_i = (l - 1) + (a_2 - l) + 1 = a_2$, a contradiction.

Suppose instead that we obtain d using the other method. Then some subset A of $\{a_3, \ldots, a_n\}$ plus the shift of $(a_1 - 1) + (a_2 - l)$ plus possibly 1 and/or l equals d, and so the sum of the elements of $S = (\{a_3, \ldots, a_n\} \backslash A)$ is -1 plus possibly 1 and/or l. Obviously -1 is impossible. The sum of the elements of S could only be zero if $a_3, \ldots, a_n$, and 1 are used, which cannot happen if $m < n$

since we are limited to using $\leq n-2$ elements from $\{1, l, a_3, \ldots, a_n\}$. Finally, suppose the sum is $l-1$ or l. Then some $a_j \leq l$ for $j \geq 3$, which is impossible since $a_j \geq a_2 \geq l+1$. Therefore $\beta_{md}^{R/L} = 0$ for $m < n$, and $\beta_{nd}^{R/L} = 1$. Thus if the Hilbert functions of L and I are the same, $i = 2$, or $a_i = a_2$. □

We can now give the proof of Proposition 4.1.

Proof: We may assume that $L = (x_1^{a_1}, \ldots, x_n^{a_n}\ x_1^{a_1-1}x_2^{a_2-l})$ for $2 \leq l \leq a_2 - 1$ and $I = (x_1^{a_1}, x_2^{a_2-1}, x_3^{a_3}, \ldots, x_n^{a_n})$. Note that $a_2 - 1 = (a_1 - 1) + (a_2 - l)$ since $a_2 - 1$ must equal the degree of the additional generator of L, so $a_1 = l$. To prove the proposition, we need to show that we can get the sum of any j of $a_1, a_2 - 1, a_3, \ldots, a_n$ as degrees of generators of the free modules in the minimal resolution of R/L. These degrees in the minimal resolution of R/L come from (1) taking the sum of j elements of $\{a_1, \ldots, a_n\}$ (but not both a_1 and a_2) or (2) taking the sum of $j-1$ elements of $\{1, a_2 - (a_2 - l) = l = a_1, a_3, \ldots, a_n\}$ (but not both 1 and a_1) plus $a_2 - 1$, the degree of the additional generator of L. (Note that we cannot take both a_1 and a_2 or 1 and a_1 because those terms will be canceled since they yield a constant in the comparison map.)

Let S be a subset of j elements of $\{a_1, a_2 - 1, a_3, \ldots, a_n\}$. If S contains a_1 and not $a_2 - 1$, we get the corresponding degree from method (1). The case in which S has neither a_1 nor $a_2 - 1$ is the same. If S contains a_1 and $a_2 - 1$, use method (2) without using the 1. Finally, if S contains $a_2 - 1$ but not a_1, we use method (2), picking the necessary $j - 1$ of $a_3, \ldots, a_n$. Therefore $\beta^{R/I} \leq \beta^{R/L}$. □

We use the information in the results above about the degrees of socle generators in LPP almost complete intersections in the following lemma.

Lemma 4.5 *Let L be an $(a_1, \ldots, a_n)-$LPP almost complete intersection. Let I be an ideal containing a regular sequence $f_1, \ldots, f_n$ in degrees $a_1 - b_1, \ldots, a_n - b_n$, with at least one $b_i > 0$, among its minimal generators. Suppose that R/I is not a complete intersection. Then I and L do not have the same Hilbert function.*

Proof: From the earlier computations, it is clear that the highest degree generator of the last term in the minimal resolution of R/L is $a_1 + \cdots + a_n - 1$. Let $F = (f_1, \ldots, f_n)$. Then the socle generator of R/F has degree $(a_1 - b_1) + \cdots + (a_n - b_n) - n$. Thus after shifting, the degree of the generator of the last term in the resolution of R/F is $(a_1 - b_1) + \cdots + (a_n - b_n)$. Since we need R/I and R/L to have the same regularity, and we are requiring some b_i to be positive, we conclude that a single $b_i = 1$, and the rest are zero. Consequently, either $H_{R/L}(r) > H_{R/F}(r)$ for some r, so I cannot have the same Hilbert function as L, or the Hilbert functions are last nonzero in the same degree s. In the latter case, both Hilbert functions are 1 in degree s. But we want R/I not to be a complete intersection, and adding another generator to F kills the socle generator of R/F, making $H_{R/I}(s)$ too small. □

Consequently, we now need only consider ideals I with a regular sequence of minimal generators in degrees $a_1, \ldots, a_n$.

5 Upper bounds on some Hilbert functions

There are two cases left to consider to prove the LPP Conjecture when L is an $(a_1, \ldots, a_n)$-LPP almost complete intersection. We need to show that if I has the same Hilbert function as L and contains a regular sequence of minimal generators in the same degrees, then

(1) I cannot have more than $n + 1$ minimal generators, and

(2) the ideal quotient (maximal regular sequence of minimal generators of I) : (other minimal generator of I) must be a complete intersection.

The next lemma proves that if either of these conditions fails, I cannot have the same Hilbert function as an LPP almost complete intersection.

Lemma 5.1 *Let $L = (x_1^{a_1}, \ldots, x_n^{a_n},\ m)$ be an $(a_1, \ldots, a_n)-$LPP almost complete intersection. Let $I = (f_1, \ldots, f_n, g)$, where $\deg f_i = a_i$, the f_i form a regular sequence, and $\deg g = \deg m$.*
(1) If $(f_1, \ldots, f_n) : (g)$ is a complete intersection, then either $H_{R/I} = H_{R/L}$, or there exists $u > 0$ such that $H_{R/I}(u) < H_{R/L}(u)$.
(2) If $(f_1, \ldots, f_n) : (g)$ is not a complete intersection, then there exists $u > 0$ such that $H_{R/I}(u) < H_{R/L}(u)$.

Proof: Write $J = (x_1^{a_1}, \ldots, x_n^{a_n})$ and $F = (f_1, \ldots, f_n)$. Let $C_L = J : (m)$, and let $C_I = F : (g)$. Write $d = \deg g = \deg m$. The canonical short exact sequences show that

$$H_{R/L}(l) = H_{R/J}(l) - H_{R/C_L}(l - d) \text{ and } H_{R/I}(l) = H_{R/F}(l) - H_{R/C_I}(l - d).$$

Of course, $H_{R/J} = H_{R/F}$. Therefore it suffices to show that if $H_{R/I} \neq H_{R/L}$, then there is an u such that $H_{R/C_I}(u) > H_{R/C_L}(u)$.

Since R/J and R/F both have a single socle generator in the same top degree, adding m and g into those ideals kills the socle element, and thus H_{R/C_I} and H_{R/C_L} are both last nonzero in the same degree.

(1) Assume first that R/C_I is a complete intersection and that $H_{R/I} \neq H_{R/L}$. Then the degrees of the minimal generators of C_I and C_L have the same sum. The degrees of the minimal generators of C_L are $1, \ldots, 1, a_{j+1} - b_{j+1}$, $a_{j+2}, \ldots, a_n$, where $0 \leq b_{j+1} < a_{j+1}$. Write the degrees of the minimal generators of C_I in weakly increasing order, and call them c_i. If the first j c_i are not all 1, we are done, for then $H_{R/C_I}(1) > H_{R/C_L}(1)$. Otherwise, let r be the first index where the list of degrees differs.

If $a_r < c_r$ (or if $r = j+1$, then if $a_{j+1} - b_{j+1} < c_{j+1}$), we are done, since then $H_{R/C_L}(a_r) < H_{R/C_I}(a_r)$. Otherwise, $a_r > c_r$ (or $a_{j+1} - b_{j+1} > c_{j+1}$). Hence there exists s such that $c_{r+s} > a_{r+s}$ because the lists of degrees have the same sum. By construction, $F \subset C_I =: (h_1, \ldots, h_n)$, where $\deg h_i = c_i$. Comparing degrees of minimal generators of F and C_I, we find

$$(f_1, \ldots, f_{r+s}) \subset (h_1, \ldots, h_{r+s-1})$$

since $\deg h_{r+s} > \deg f_l$ for $l \leq r+s$ (recall that the a_i, the degrees of the f_i, are weakly increasing). But this means that a depth $r+s$ ideal is contained in an ideal of smaller depth, a contradiction.

Consequently, if R/C_I is a complete intersection, we cannot have $H_{R/C_I}(l) \leq H_{R/C_L}(l)$ for each l without equality occurring for every l. This gives the desired inequality for $H_{R/I}$ and $H_{R/L}$.

(2) Suppose instead that R/C_I is not a complete intersection but that $H_{R/C_I}(l) \leq H_{R/C_L}(l)$ for all l. C_I must have at least as many linear generators as C_L, or we are done. Moreover, since $F \subset C_I$, there is an $(a_1, \dots, a_n)$ complete intersection inside C_I.

As before, the regularities of C_I and C_L are the same, and so the sum of the degrees of any maximal length regular sequence inside C_I must exceed, in the notation above, $j + (a_{j+1} - b_{j+1}) + a_{j+2} + \cdots + a_n$. We form a minimal generating set of C_I as follows: Pick a linear form inside C_I, and call it h_1. Now pick another element of C_I of minimal degree, calling it h_2, such that h_1 and h_2 form a regular sequence. Continue until one has a regular sequence $h_1, \dots, h_n$ of length n with $\deg h_i \leq \deg h_{i+1}$. Write $C_I = (h_1, \dots, h_n, h_{n+1}, \dots, h_p)$, where $p \geq n+2$ (since a theorem of Kunz shows that almost complete intersections are never Gorenstein [13]).

Let us consider the degrees of these generators. We know $\deg h_i = 1$ for $i \leq j$. Let $c_s = \deg h_s$ for $j+1 \leq s \leq n$. Then

$$c_{j+1} + \cdots + c_n > (a_{j+1} - b_{j+1}) + a_{j+2} + \cdots + a_n.$$

If $a_{j+1} - b_{j+1} < c_{j+1}$, then we are done, for then $H_{R/C_L}(a_{j+1} - b_{j+1}) < H_{R/C_I}(a_{j+1} - b_{j+1})$. Otherwise, $a_{j+1} - b_{j+1} \geq c_{j+1}$. Hence there exists $r > j+1$ such that $a_r < c_r$ because of the restriction on the sums of the degrees. Note that $(f_1, \dots, f_r)$ must be contained in the portion of C_I in degrees at most a_r. Therefore, if $\{h_{q_1}, \dots, h_{q_t}\}$ is the subset of $\{h_{n+1}, \dots, h_p\}$ of polynomials of degree at most a_r, we have

$$(f_1, \dots, f_r) \subset (h_1, \dots, h_{r-1}, h_{q_1}, \dots, h_{q_t}).$$

But the left-hand side has depth r, while our construction guarantees that the right-hand side has depth only $r-1$, a contradiction. Hence there must be some l such that $H_{R/C_I}(l) > H_{R/C_L}(l)$, and we are done. □

Analyzing the preceding proof, we obtain a special case of Conjecture 1.2.

Corollary 5.2 *Let $a_1 \leq \cdots \leq a_n$ be positive integers. Let I be an almost complete intersection generated by homogeneous polynomials $f_1, \dots, f_n$, and g, where the f_i form a regular sequence, $\deg f_i = a_i$, and $\deg g = d$. Assume $d \geq a_1$. Let $L = (x_1^{a_1}, \dots, x_n^{a_n}, m)$, where m is the greatest monomial in lex order in degree d not in $(x_1^{a_1}, \dots, x_n^{a_n})$. Then $H_{R/I}(d+1) \leq H_{R/L}(d+1)$.*

Given an ideal I as above, the idea is to take the appropriate LPP almost complete intersection L so that $H_{R/I}(d) = H_{R/L}(d)$, where d is the degree of the extra generator g of I. Then we show that the Hilbert function of R/I cannot grow any faster than that of R/L in the next degree. Our assumption that $d \geq a_1$ is not actually a significant restriction; one can choose the generators of the maximal regular sequence to ensure that this is the case.

Proof: The proof is almost identical to that of Lemma 5.1. We outline the case in which $(f_1, \ldots, f_n) : (g) = (h_1, \ldots, h_n)$ is a complete intersection. In the notation of Lemma 5.1, we need to show that $H_{R/C_I}(1) \geq H_{R/C_L}(1)$. If not, then there are more minimal linear generators in C_I than in C_L. Since the regularity of the ideals is the same, there is some $r > j+1$ such that $\deg h_r > a_r$, where we have ordered the h_i such that $\deg h_i \leq \deg h_{i+1}$ as above. Then $(f_1, \ldots, f_r) \subset (h_1, \ldots, h_{r-1})$, a contradiction. The proof of the case in which $(f_1, \ldots, f_n) : (g)$ is not a complete intersection is also essentially identical to the argument in Lemma 5.1. □

To conclude this section, we state two more results of this nature.

Proposition 5.3 *Under the hypotheses of Corollary 5.2:*
(1) If $H_{R/I} \neq H_{R/L}$, let l_0 be the first degree such that $H_{R/I}(l) \neq H_{R/L}(l)$. Then $H_{R/I}(l_0) < H_{R/L}(l_0)$.
(2) If $(f_1, \ldots, f_n) : g$ is a complete intersection, then $H_{R/I}(l) \leq H_{R/L}(l)$ for all l. □

Proof: (1) This is very similar to the proofs of the previous results. Suppose $(f_1, \ldots, f_n) : (g)$ is not a complete intersection (if it is, then (1) follows from (2)). As in Corollary 5.2, using the same notation, C_I has at least as many linear generators as C_L; if it has fewer, then $H_{R/I}(d+1) < H_{R/L}(d+1)$, and we are done. Otherwise, choose a maximal length regular sequence $h_1, \ldots, h_n$ inside C_I as before, picking something of minimal degree each time. Suppose C_L and C_I have j linear generators. If $\deg h_{j+1} > a_{j+1} - b_{j+1}$, we are done, for then $H_{R/I}(d + a_{j+1} - b_{j+1}) < H_{R/L}(d + a_{j+1} - b_{j+1})$. If not, then since the sum of the degrees of $h_1, \ldots, h_n$ is greater than the sum of the degrees of the minimal generators of C_L, for some $n \geq i > j+1$, $\deg h_i > a_i$. But then, as before, we have an ideal $(f_1, \ldots, f_i)$ of depth i contained in an ideal of depth $i-1$, a contradiction.

(2) Let C_I and C_L be as before. Call the minimal generators of C_I $h_1, \ldots, h_n$, with their degrees weakly increasing. We need to show that $H_{R/C_I}(l) \geq H_{R/C_L}(l)$ for all l. To do this, we analyze how changing the degrees of the minimal generators of a complete intersection in a particular way affects the Hilbert series.

We need to determine the possible degrees $v_l = (q_1, \ldots, q_n)$ of the minimal generators of C_I, where $q_i \leq q_{i+1}$ for each i. By previous arguments, we know that $q_i \leq a_i$ for all i. We claim that we can start with the degree vector of the minimal generators of C_L, $v_0 = (1, \ldots, 1, a_{j+1} - b_{j+1}, a_{j+2}, \ldots, a_n)$, and move to any possible degree vector $(q_1, \ldots, q_n)$ of the minimal generators of C_I by a sequence of switches of degrees from some c_j and c_i to $c_j - 1$ and $c_i + 1$, where

$c_j > c_i + 1$. (Note that the regularities of C_I and C_L are the same, so the sums of the degrees of the minimal generators of each are the same.) To do this, we proceed in the following manner. Starting with v_0, take the greatest entry of v_0 that is greater than its corresponding entry in v_l. Subtract one from it and add that to the lowest entry of v_0 that needs to increase; then sort so that the new list is weakly increasing. Here, the ones in v_0 cannot decrease, and the entries of a_i cannot increase, so we do not have to subtract from a lesser degree and add to a greater number.

Suppose we get to some $v_c = (c_1, \dots, c_n)$ in the algorithm, where $c_i \leq c_{i+1}$ for each i. Say c_j is the greatest number that needs to decrease and c_i is the least number that needs to increase. We show that $c_j > c_i + 1$. If $c_j = c_i + 1$, then we are not changing the degrees by taking one from c_j and adding it to c_i. This is not a move the algorithm can require us to make because in this case, $q_i > c_i = c_j - 1 \geq q_j$, a contradiction since $j > i$. If $c_j < c_i + 1$, then we previously needed an entry of $c_i + 1$ (or greater), and so we should not have decreased c_j (or something greater) earlier.

For example, we might start with degrees $(1, 1, 1, 5, 7, 9)$ and need to switch to $(2, 2, 4, 6, 8)$. We would do this by changing $(1, 1, 1, 5, 7, 9)$ to $(1, 1, 2, 5, 7, 8)$ to $(1, 2, 2, 5, 6, 8)$ to $(2, 2, 2, 4, 6, 8)$.

Now, let us compute the effect of such switches in the degrees of minimal generators of complete intersection ideals on the Hilbert function. For ease in notation, without loss of generality, suppose we change a complete intersection with minimal generators in degrees $c_1, \dots, c_n$ to one with minimal generators in degrees $c_1 + 1, c_2 - 1, c_3, \dots, c_n$. We may assume by the arguments above that $c_2 - c_1 - 1 > 0$. We compute the Hilbert series of $R/(\text{new ideal})$ minus the Hilbert series of $R/(\text{old ideal})$, which is:

$$\frac{(1 - t^{c_1+1})(1 - t^{c_2-1})\prod_{i=3}^{n}(1 - t^{c_i})}{(1-t)^n} - \frac{(1 - t^{c_1})(1 - t^{c_2})\prod_{i=3}^{n}(1 - t^{c_i})}{(1-t)^n} =$$

$$\frac{\prod_{i=3}^{n}(1 - t^{c_i})}{(1-t)^n}\left(t^{c_1} + t^{c_2} - t^{c_2-1} - t^{c_1+1}\right) = \frac{\prod_{i=3}^{n}(1 - t^{c_i})}{(1-t)^n} t^{c_1}\left(1 - t\right)\left(1 - t^{c_2-c_1-1}\right)$$

$$= t_1^c \left(\frac{1 - t^{c_2-c_1-1}}{1-t}\right)\left(\frac{1-t}{1-t}\right)\left(\frac{1-t^{c_3}}{1-t}\right)\cdots\left(\frac{1-t^{c_n}}{1-t}\right).$$

Since $c_2 - c_1 - 1 > 0$, this expression is a polynomial with all nonnegative coefficients, so the Hilbert function of R/C_L is at most that of R/C_I in every degree. Hence $H_{R/I}(l) \leq H_{R/L}(l)$ for all r. □

We believe that the stronger statement of (2) in Proposition 5.3 should also hold when $(f_1, \dots, f_n) : g$ is not a complete intersection. Finally, we note that if $f_1, \ldots, f_n$ and g are generic, then the inequality $H_{R/I}(l) \leq H_{R/L}(l)$ holds for all l by a result of Fröberg in [7]; one can also compute this directly without much difficulty.

6 Conclusion

We can now give our main result.

Theorem 6.1 *Let L be an $(a_1, \ldots, a_n)-$LPP almost complete intersection. Let I be any ideal with the same Hilbert function as L that contains a regular sequence in degrees $a_1, \ldots, a_n$. Then $\beta^{R/I} \leq \beta^{R/L}$.*

Proof: If I contains a complete intersection in degrees less than $a_1, \ldots, a_n$, then by the results of Section 4, I is an $(a_1, a_2 - 1, a_3, \ldots, a_n)$ complete intersection. The theorem then follows from Proposition 4.1. Otherwise, I has minimal generators $f_1, \ldots, f_n$ that form a regular sequence in degrees $a_1, \ldots, a_n$. By Lemma 5.1, I is an almost complete intersection $(f_1, \ldots, f_n, g)$ with $(f_1, \ldots, f_n) : (g)$ a complete intersection. Then Theorem 3.3 gives the inequality for the Betti numbers. □

The extra structure of almost complete intersections is important in our arguments. All the mapping cones we have used come from two Koszul complexes, and one loses this structure if one has to consider ideals with $n+2$ or more generators. In particular, it becomes much more difficult to detect nonminimality.

There has been some other recent work in this area. Gasharov, Hibi, and Peeva have introduced the idea of **a**-stable ideals, and they study their homological properties in [9]. Richert [18] has proven the LPP Conjecture (and Conjecture 1.2) in the case $n = 2$ and for monomial ideals in three variables. He also has some reductions that we hope will make it possible to understand connections between the Eisenbud-Green-Harris and LPP Conjectures more clearly in the future.

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