

# An Invitation to Geometric Quantization

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# What is quantization?

Quantization is a process of associating a classical mechanical system to a Hilbert space. Through this process, classical observables are sent to linear operators on the Hilbert space.

## Example

Particle moving on  $\mathbb{R}^1$ . The configuration space is the space of all possible positions of the particle, which is  $\mathbb{R}^1$ . The phase space is

$$T^*\mathbb{R} = \{(q, p) \mid q \in \mathbb{R}^1, p \in T_q^*\mathbb{R}^1\}$$

where  $q$  is the position and  $p$  is the momentum.

# What is quantization?

## Example

Suppose the particle is subject to a potential energy which depends on  $q$  (an example is the simple harmonic oscillator). Then

$$\frac{p^2}{2m} + V(q) = \text{constant}$$

Turning the crank of quantization,

- $\mathcal{H} = L^2(\mathbb{R}^1)$
- $q \mapsto M_x = \text{Multiplication by } x$
- $p \mapsto -i\hbar \frac{d}{dx}$
- $\frac{p^2}{2m} + V(q) \mapsto -\frac{\hbar^2}{2m} \Delta + M_V$  (Schrödinger operator)

# Symplectic manifolds

## Definition

$(X, \omega)$  is a *symplectic manifold* if

- $X$  is a manifold
- $\omega$  is a closed, non-degenerate 2-form

A compact symplectic manifold  $(X, \omega)$  plays the rôle of a classical phase space.

# Symplectic manifolds

## Example

$(X, \omega) = (S^2, \text{area form})$ .

$$\omega_p(\xi, \eta) = \langle \xi \times \eta, \hat{n}_p \rangle$$

Two questions:

- Can we always quantize any  $(X, \omega)$ ?
- What is the corresponding Hilbert space?

# First attempt

We want the Hilbert space to be a certain space of sections of a complex line bundle  $L$  on  $X$  equipped with a connection  $\nabla$  and a covariant inner product  $\langle \cdot, \cdot \rangle$  such that

$$\text{curv}(\nabla) = \omega$$

So this imposes a condition on  $\omega$  already.

## Proposition

$[\omega] \in H^2(X, \mathbb{Z})$  iff there exists  $(L, \nabla, \langle \cdot, \cdot \rangle)$  such that  $\text{curv}(\nabla) = \omega$ .

# First attempt

## Definition

$(X, \omega)$  is *prequantizable* if  $[\omega] \in H^2(X, \mathbb{Z})$ .

## Remark

$[\omega] = c_1(L)$ . *The class of  $\omega$  is topological in nature and does not depend on  $\nabla$ .*

# First attempt

## Definition

$(L, \nabla, \langle, \rangle)$  are prequantum data of  $(X, \omega)$  if

- $\text{curv}(\nabla) = \omega$
- $\langle, \rangle$  is covariant under  $\nabla$

The Hilbert space is

$$\mathcal{H} = \left\{ s \in \Gamma(L) \mid \int_X \langle s, s \rangle \frac{\omega^n}{n!} < +\infty \right\}$$

# First attempt

Given a function  $f \in C^\infty(X)$ , what is the associated operator  $Q_f$ ?

## Definition

$X_f$  is the symplectic vector field such that

$$\iota_{X_f}\omega = df$$

## Definition

$$Q_f = \nabla_{X_f} + if$$

# First attempt

## Proposition

$Q_f$  is skew-adjoint with respect to the inner product  $\langle\langle \cdot, \cdot \rangle\rangle$  on  $\Gamma(L)$  defined by

$$\langle\langle s, s' \rangle\rangle = \int_X \langle s, s' \rangle \frac{\omega^n}{n!}$$

# A classical example

## Example

Let  $X = S^2$  the unit sphere in  $\mathbb{R}^3$  centered at the origin,  
 $\omega =$  Area form. If

- $L = TS^2$
- $\nabla =$  Riemannian connection induced from that of  $T\mathbb{R}^3$
- $\langle, \rangle =$  Riemannian metric

Then  $(L, \nabla, \langle, \rangle)$  are prequantum data.

# A classical example

## Example

We have

- $X_f =$  Tangent vectors of latitudes
- More precisely, if  $p = (\varphi, \theta)$  in spherical coordinates,  
 $(X_f)_p = \sin \varphi (\cos \theta \vec{i} + \sin \theta \vec{j})$
- $Q_f X_f = 0$

# Disadvantage of first attempt

$\mathcal{H}$  obtained from this quantization scheme is too large to handle.

One way to get around this is to introduce *polarization* and holomorphic sections to cut down the dimensions of  $\mathcal{H}$ . Then we need to impose more structures on the compact symplectic manifold.

A natural candidate: Kähler manifold.

# Kähler manifolds

## Definition

$(X, \omega, J)$  is a *Kähler manifold* if

- $\omega$  is a symplectic 2-form.
- $J$  is an integrable almost complex structure, i.e.  $X$  is a complex manifold and  $J$  corresponds to multiplication by  $i$  on each fiber of  $T\mathcal{U}$  where  $\mathcal{U}$  is a holomorphic chart.
- $\omega$  and  $J$  are compatible in the sense that  $\omega(\cdot, J\cdot)$  is positive definite.

# Examples of Kähler manifolds

## Example

$X = \mathbb{C}^n = \{(x_1 + \sqrt{-1}y_1, \dots, x_n + \sqrt{-1}y_n) \mid x_i, y_i \in \mathbb{R}, 1 \leq i \leq n\}$ .  
Identifying  $T_p\mathbb{C}^n$  with  $\mathbb{C}^n$ , letting  $e_i$  be the  $i$ -th standard basis vector and  $f_i = \sqrt{-1}e_i$ , we define  $J$  by

$$Je_i = f_i, \quad Jf_i = -e_i$$

and

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i$$

Then  $(\mathbb{C}^n, \omega, J)$  is a Kähler manifold. Actually any Kähler manifold locally looks like  $(\mathbb{C}^n, \omega, J)$ .

# Examples of Kähler manifolds

## Example

$(S^2, \omega, J)$ , where  $\omega =$  area form and  $J$  is the rotation by  $\frac{\pi}{2}$  counterclockwise on tangent spaces when viewing  $S^2$  from outside, is Kähler.

## Example

$\mathbb{C}P^n$  and any smooth projective subvarieties of  $\mathbb{C}P^n$  are Kähler.

# Examples of Kähler manifolds

## Example

The coadjoint action of a compact Lie group  $G$  on  $\mathfrak{g}^*$  is defined by

$$\langle \mathrm{Ad}_g^* \gamma, \xi \rangle = \langle \gamma, \mathrm{Ad}_{g^{-1}} \xi \rangle$$

Let  $\mathcal{O}_\gamma$  be the orbit of the coadjoint action of  $G$  passing through  $\gamma \in \mathrm{Int}(\Lambda_+^*)$ . Then

- $\mathcal{O}_\gamma \cong G/T$ ,  $T$  being a maximal torus of  $G$ .
- $T_\beta \mathcal{O}_\gamma \cong \mathfrak{g}/\mathfrak{t}$ .
- Using the above identifications, we define a 2-form

$$\omega_\beta(\xi, \eta) = \beta([\xi, \eta])$$

$\omega$ , called the Kostant-Kirillov-Souriau form, is symplectic and integral.

# Examples of Kähler manifolds

## Example

Consider the complexified Lie algebra  $\mathfrak{g}_{\mathbb{C}}$  and its root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}$$

Let  $\{H_{\alpha}, X_{\alpha}\}_{\alpha \in R}$  be the Chevalley basis of  $\mathfrak{g}_{\mathbb{C}}$  which satisfies

- $[H_{\alpha}, X_{\beta}] = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} X_{\beta}$
- $[X_{\alpha}, X_{-\alpha}] = H_{\alpha}$  for  $\alpha \in R^{+}$

Then

$$\mathfrak{g}/\mathfrak{t} = \bigoplus_{\alpha \in R^{+}} \text{span}_{\mathbb{R}}\{e_{\alpha}, f_{\alpha}\}$$

where  $e_{\alpha} = \sqrt{-1}(X_{\alpha} + X_{-\alpha})$ ,  $f_{\alpha} = X_{\alpha} - X_{-\alpha}$ .

# Examples of Kähler manifolds

## Example

Define  $J$  by

$$Je_\alpha = f_\alpha, \quad Jf_\alpha = -e_\alpha$$

Then  $(\mathcal{O}_\gamma, \omega, J)$  is Kähler.

# Kähler polarization

Suppose  $\dim_{\mathbb{C}} X = n$ . Consider the complexified tangent bundle  $TX \otimes_{\mathbb{R}} \mathbb{C}$

## Definition

A complex rank  $n$  subbundle  $F \subseteq TX \otimes_{\mathbb{R}} \mathbb{C}$  is a *positive-definite polarization* if

- It is integrable, i.e. closed under Lie bracket.
- For all  $X, Y \in F$ ,  $\omega_{\mathbb{C}}(X, Y) = 0$
- $\sqrt{-1}\omega_{\mathbb{C}}(\cdot, \bar{\cdot})$  is positive-definite.

# Kähler polarization

## Example

$X = \mathbb{C}^n$ . Then

$$T_p X \otimes_{\mathbb{R}} \mathbb{C} \cong \text{span}_{\mathbb{C}}\{e_1, f_1, \dots, e_n, f_n\}$$

Then

$$F = \text{span}_{\mathbb{C}}\{e_1 + \sqrt{-1}f_1, \dots, e_n + \sqrt{-1}f_n\}$$

is a positive-definite polarization.

$F$  should be thought of as the 'holomorphic direction' in  $TX \otimes_{\mathbb{R}} \mathbb{C}$ .

## Second attempt: Kähler quantization

### Theorem

Let  $(X, \omega, J)$  be a compact Kähler manifold with positive-definite polarization  $F$ , and  $(L, \nabla, \langle, \rangle)$  be prequantum data. Let

$$X_{\text{quantum}} = \{s \in \Gamma(L) \mid \nabla_{\Theta} s = 0 \text{ for all } \Theta \in \bar{F}\}$$

Then  $X_{\text{quantum}}$  is finite-dimensional.

We define the quantization of  $(X, \omega, J)$  to be  $X_{\text{quantum}}$ , which is the space of holomorphic sections of the prequantum line bundle  $L$ .

## Second attempt: Kähler quantization

### Example

$X = (\mathcal{O}_\gamma, \omega, J)$  as in a previous example. A positive-definite polarization is

$$F = \text{span}_{\mathbb{C}}\{e_\alpha + \sqrt{-1}f_\alpha\}_{\alpha \in R^+} = \text{span}_{\mathbb{C}}\{X_\alpha\}_{\alpha \in R^+}$$

So  $\bar{F} = \text{span}_{\mathbb{C}}\{X_{-\alpha}\}_{\alpha \in R^+}$ . Note that  $L = G \times^T \mathbb{C}_\gamma$  is a prequantum line bundle. By Borel-Weil Theorem,

$$\begin{aligned} X_{\text{quantum}} &= \text{space of holomorphic sections of } L \\ &= \text{Irreducible representation of } G \text{ with highest weight } \gamma \end{aligned}$$

## Second attempt: Kähler quantization

### Example

$X = (S^2, \omega, J)$  with  $L = TS^2$ . Then  $X_{\text{quantum}}$  is a 3-dimensional complex vector space. Identifying  $S^2$  with  $\mathbb{C} \cup \infty$  through stereographic projection, we can describe three holomorphic vector fields of  $S^2$  which form a basis of  $X_{\text{quantum}}$  as follows

- the vector field  $s_0$  generated by the infinitesimal action of  $1 \in \text{Lie}(S^1)$  of the  $S^1$ -action on  $\mathbb{C} \cup \infty$  given by rotation  $z \mapsto e^{i\theta}z$ ,
- the vector field  $s_{-2}$  generated by the infinitesimal action of  $1 \in \text{Lie}(\mathbb{R}^1)$  of the  $\mathbb{R}^1$ -action on  $\mathbb{C} \cup \infty$  given by translation  $z \mapsto z + a$ ,
- the vector field  $s_2$  generated by the infinitesimal action of  $1 \in \text{Lie}(\mathbb{R}^1)$  of the  $\mathbb{R}^1$ -action on  $\mathbb{C} \cup \infty$  given by  $z \mapsto \frac{1}{z + a}$

# Quantization of $G$ -Kähler manifolds

One may further consider a compact Kähler manifold with prequantum data and a nice  $G$ -action where  $G$  is a compact Lie group. By nice we mean

- $G$  preserves both  $\omega$  and  $J$ .
- There exists a map called *moment map*

$$\mu : X \rightarrow \mathfrak{g}^*$$

which is  $G$ -equivariant (here  $G$  acts on  $\mathfrak{g}^*$  by coadjoint action) and

$$\iota_{\xi\#}\omega = d\langle\mu, \xi\rangle \quad \text{for all } \xi \in \mathfrak{g}$$

We say  $G$  acts on  $X$  in a Hamiltonian fashion.

# Quantization of $G$ -Kähler manifolds

Let

$$Q_\xi := Q_{\langle \mu, \xi \rangle} = \nabla_{\xi^\#} + i\langle \mu, \xi \rangle$$

This gives a  $\mathfrak{g}$ -action on  $\Gamma(L)$ . In nice cases, e.g.  $G$  is simply-connected, this action can be integrated to a  $G$ -action. It turns out that  $G$  acts on  $X_{\text{quantum}}$ , which makes it a  $G$ -representation.

Question: What can we say about the multiplicities of weights of this representation?

# Quantization commutes with reduction

## Definition

Let  $G$  act on  $(X, \omega, J)$  in a Hamiltonian fashion with moment map  $\mu$ . Assume that  $0$  is a regular value of  $\mu$  and  $G$  acts on  $\mu^{-1}(0)$  freely. The *symplectic reduction* of  $X$  by  $G$  is defined to be

$$X_G := \mu^{-1}(0)/G$$

$X_G$  can be thought of as the fixed 'points' of the phase space  $X$ .

One can construct prequantum data  $(L_G, \nabla_G, \langle, \rangle_G)$  and positive-definite polarization of  $X_G$  from those of  $X$  by restriction and quotienting.

# Quantization commutes with reduction

## Theorem (Guillemin-Sternberg, '82)

$$\dim(X_{\text{quantum}}^G) = \dim((X_G)_{\text{quantum}})$$

By virtue of this theorem, one can compute the multiplicity of the trivial representation in  $X_{\text{quantum}}$  by looking at the quantization of  $X_G$ .

# Quantization commutes with reduction

## Example

$X = (S^2, \omega, J)$ ,  $G = S^1$  acts on  $X$  by rotation around the z-axis, with  $\mu$  being the height function. Note that

$$e^{i\theta} \cdot s_0 = s_0, \quad e^{i\theta} \cdot s_{-2} = e^{-2i\theta} s_{-2}, \quad e^{i\theta} \cdot s_2 = e^{2i\theta} s_2$$

So as  $S^1$ -representations,

$$X_{\text{quantum}} \cong \mathbb{C}_0 \oplus \mathbb{C}_{-2} \oplus \mathbb{C}_2$$

It follows that  $\dim(X_{\text{quantum}}^{S^1}) = 1$ . On the other hand,

$$X_{S^1} = \text{a point}$$

A line bundle over a point is simply a 1-dimensional vector space. So  $\dim((X_{S^1})_{\text{quantum}}) = 1$ .