# MATH1120 Calculus II

Notes 10
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## 1. Infinite series

**Theorem 1.1.** *If $a_n \geq 0$ for any $n$, then both $\sum_{n=1}^{\infty} a_n$ and the series obtained by adding brackets to, removing brackets from, and rearranging terms of $\sum_{n=1}^{\infty} a_n$ converge or both diverge. If both converge, their sums are the same.*

**Exercise 1.1.** Find $\sum_{n=1}^{\infty} \frac{n}{8^n}$.

## 2. Convergence tests for infinite series with nonnegative terms

There are various convergence tests for infinite series with nonnegative terms. The common idea behind these tests lies in comparison of the given infinite series with other infinite series(or improper integrals) with known convergence property.

### 2.1. Partial sums with an upper bound.

**Theorem 2.1.** *If the sequence $\{a_n\}_{n=1}^{\infty}$ is nondecreasing and there exists $M$ such that $a_n \leq M$ for all $n$, then $\{a_n\}_{n=1}^{\infty}$ is convergent. If $\{a_n\}_{n=1}^{\infty}$ is unbounded, then it is divergent.*

Consider a series $\sum_{n=1}^{\infty} a_n$ whose terms are nonnegative. So the partial sums $s_n$ form a nondecreasing sequence(because $s_{n+1} = s_n + a_{a+1} \geq s_n$). It follows from the above theorem that

**Corollary 2.2.** *A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges iff there exists $M$ such that $s_n \leq M$ for all $n$.*

**Example 2.3.** The *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges because the sequence of partial sums is unbounded. One can see this by grouping the terms as follows

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

The sum in each parenthesis is greater than $\frac{1}{2}$. So $s_{2^k} > \frac{k}{2}$.

2.2. **The Integral Test.**

**Theorem 2.4.** *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative terms. Suppose that* $a_n = f(n)$*, where* $f$ *is continuous, nonnegative decreasing function for all* $x \geq N$ *for some natural number* $N$*. Then* $\sum_{n=1}^{\infty} a_n$ *and* $\int_N^{\infty} f(x)dx$ *both converge or both diverge.*

**Theorem 2.5** (Error estimate)**.** *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of nonnegative terms. Suppose that* $a_n = f(n)$*, where* $f$ *is continuous, nonnegative decreasing function for all* $x \geq N$ *for some natural number* $N$*, and* $\sum_{n=1}^{\infty} a_n$ *converges and is equal to* $S$*. Then*

$$\int_{N+1}^{\infty} f(x)dx \leq S - s_N \leq \int_N^{\infty} f(x)dx$$

**Example 2.6.** By the integral test, the *p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$. The $p$-series are commonly used in comparison tests.

**Exercise 2.1.** Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$.

**Exercise 2.2.** Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$ converges.

2.3. **The Direct Comparison Test.**

**Theorem 2.7.** $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are infinite series with nonnegative terms.*

*(1) If* $a_n \leq b_n$ *for* $n \geq N$ *for some natural number* $N$*, and* $\sum b_n$ *converges, then so does* $\sum a_n$*.*

*(2) If* $a_n \leq b_n$ *for* $n \geq N$ *for some natural number* $N$*, and* $\sum a_n$ *diverges, then so does* $\sum b_n$*.*

**Remark 2.8.** When using the Direct Comparison Test, the following facts are useful.

*As* $n \to \infty$*, the growth rate of* ln *functions are always slower than that of polynomial functions(power functions), which in turn are always slower than that of exponential functions.*

*Sines and Cosines are always bounded between -1 and 1.*

**Example 2.9.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ converges because $\displaystyle\frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

**Example 2.10.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\ln n}$ diverges because $\displaystyle\frac{1}{\sqrt{n}\ln n} \geq \frac{1}{\sqrt{n}\sqrt{n}} = \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

**Exercise 2.3.** Determine the convergence of

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{\frac{3}{2}}}$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 30}$

2.4. **The Limit Comparison Test.**

**Theorem 2.11.** $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *are infinite series with positive terms.*

*(1) If* $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, *then* $\sum a_n$ *and* $\sum b_n$ *both converge or both diverge.*

*(2) If* $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*

*(3) if* $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum a_n$ *diverges, then* $\sum b_n$ *diverges.*

**Remark 2.12.** (1) The comparison series are often chosen to be those whose convergence is known, e.g. the $p$-series.

(2) When using the Limit Comparison Test, the comparison series can be obtained by picking the dominant terms(e.g. highest order terms in a polynomial). For example, for $\displaystyle\sum_{n=1}^{\infty} \frac{5n^3 - 3n}{n^2(n-2)(n^2+5)}$, a possible comparison series is $\displaystyle\sum_{n=1}^{\infty} \frac{5n^3}{n^2 \cdot n \cdot n^2} = \sum_{n=1}^{\infty} \frac{5}{n^2}$.

**Exercise 2.4.** Determine the convergence of the following series

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$

(2) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$. (*Hint*: Limit Comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$)

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + 2 + \cdot + n}$. Can you find the sum if the series converges? (*Hint*: $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$).

(4) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$. (*Hint*: compare with one $p$-series).

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{(n^3+1)\ln n}$ (*Hint*: you need to use both comparison tests).