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ORTHOGONAL MEASURES AND ERGODICITY

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We establish a strengthening of the $\mathbb{E}_0$ dichotomy analogous to the known strengthening of the perfect set theorem for families of pairwise orthogonal Borel probability measures. We then use this to characterize the class of countable Borel equivalence relations admitting ergodic Borel probability measures which are not strongly ergodic.

A *Polish space* is a separable topological space admitting a compatible complete metric. A subset of such a space is K_σ if it is a countable union of compact sets, and *Borel* if it is in the σ-algebra generated by the underlying topology. A function between such spaces is *Borel* if pre-images of open sets are Borel.

A *Borel measure* on such a space is a function $\mu \colon \mathscr{B} \to [0, \infty]$, where $\mathscr{B}$ denotes the family of Borel subsets of X, such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \mu(B_n)$ whenever $(B_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint Borel subsets of X. A *Borel probability measure* on X is a Borel measure μ on X for which $\mu(X) = 1$. We use $P(X)$ to denote the space of all Borel probability measures on X, equipped with the Polish topology generated by the functions of the form $\mu \mapsto \int f \, d\mu$, where f varies over all bounded continuous functions $f \colon X \to \mathbb{R}$ (see, for example, [Kec95, Theorem 17.23]).

A Borel set $B \subseteq X$ is *μ-null* if $\mu(B) = 0$, and *μ-conull* if its complement is μ-null. Two Borel measures μ and ν on X are *orthogonal*, or $\mu \perp \nu$, if there is a Borel set $B \subseteq X$ which is μ-conull and ν-null. More generally, we say that a sequence $(\mu_i)_{i \in I}$ of Borel measures on X is *orthogonal* if there is a sequence $(B_i)_{i \in I}$ of pairwise disjoint Borel subsets of X such that B_i is μ_i-conull for all $i \in I$. In this case, we say that $(B_i)_{i \in I}$ is a *witness* to the orthogonality of $(\mu_i)_{i \in I}$.

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When I is a Polish space, we say that a sequence $(\mu_i)_{i \in I}$ of Borel probability measures on X is *continuous* if it is continuous when viewed as a function from I to $P(X)$. We say that a sequence $(K_i)_{i \in I}$ of subsets of X is *compact* if the corresponding set $K = \{(i, x) \in I \times X \mid x \in K_i\}$ is compact. Similarly, we say that $(K_i)_{i \in I}$ is K_σ if the corresponding set $K \subseteq I \times X$ is K_σ.

Burgess-Mauldin have essentially shown that if X is a Polish space and $(\mu_c)_{c \in 2^\mathbb{N}}$ is a continuous sequence of pairwise orthogonal Borel probability measures on X, then there is a continuous injection $\phi \colon 2^\mathbb{N} \to 2^\mathbb{N}$ for which the continuous subsequence $(\mu_{\phi(c)})_{c \in 2^\mathbb{N}}$ admits a K_σ witness to orthogonality (see [BM81, Theorem 1]). Our first goal is to provide an analogous result in which $2^\mathbb{N}$ is replaced with $2^\mathbb{N} / \mathbb{E}_0$, where $\mathbb{E}_0$ is the equivalence relation on $2^\mathbb{N}$ given by

$$c \mathbin{\mathbb{E}_0} d \iff \exists n \in \mathbb{N} \forall m \geq n \; c(m) = d(m).$$

We say that Borel measures μ and ν on X are *(measure) equivalent*, or $\mu \sim \nu$, if they have the same null Borel sets. We say that measure-equivalence classes M and N of Borel measures on X are *orthogonal*, or $M \perp N$, if $\mu \perp \nu$ for some (equivalently, all) $\mu \in M$ and $\nu \in N$. More generally, we say that a sequence $(M_i)_{i \in I}$ of measure-equivalence classes of Borel measures on X is *orthogonal* if there is a sequence $(B_i)_{i \in I}$ of pairwise disjoint Borel subsets of X such that B_i is μ-conull for some (equivalently, all) $\mu \in M_i$, for all $i \in I$. In this case, we again say that $(B_i)_{i \in I}$ is a *witness* to the orthogonality of $(M_i)_{i \in I}$.

When I is a Polish space and E is an equivalence relation on I, we say that a sequence $(M_C)_{C \in I/E}$ of measure-equivalence classes of Borel probability measures on X is *continuous* if there is a continuous sequence $(\mu_i)_{i \in I}$ of Borel probability measures on X with the property that $\mu_i \in M_{[i]_E}$ for all $i \in I$. We say that a sequence $(K_C)_{C \in I/E}$ of subsets of X is *compact* if the sequence $(K_i)_{i \in I}$ given by $K_i = K_{[i]_E}$ is compact. Similarly, we say that $(K_C)_{C \in I/E}$ is K_σ if the corresponding sequence $(K_i)_{i \in I}$ is K_σ. We say that a function $\phi \colon X/E \to Y/F$ is *continuous* if there is a continuous function $\psi \colon X \to Y$ such that $\psi(x) \in \phi([x]_E)$ for all $x \in X$.

Theorem 1. *Suppose that X is a Polish space and $(M_C)_{C \in 2^\mathbb{N}/\mathbb{E}_0}$ is a continuous sequence of pairwise orthogonal measure-equivalence classes of Borel probability measures on X. Then there is a continuous injection $\phi \colon 2^\mathbb{N}/\mathbb{E}_0 \to 2^\mathbb{N}/\mathbb{E}_0$ for which the corresponding continuous subsequence $(M_{\phi(C)})_{C \in 2^\mathbb{N}/\mathbb{E}_0}$ admits a K_σ witness to orthogonality.*

Proof. Fix a continuous sequence $(\mu_c)_{c \in 2^\mathbb{N}}$ such that $\mu_c \in M_{[c]_{\mathbb{E}_0}}$ for all $c \in 2^\mathbb{N}$, as well as positive real numbers δ_n and ϵ_n with $\lim_{n \to \infty} \delta_n = 0$ and

$\sum_{n \in \mathbb{N}} \epsilon_n < \infty$. We will recursively construct natural numbers $k_n > 0$ and sets $U_{n,i} \subseteq X$, expressible as finite unions of open sets of diameter at most δ_n, such that $\overline{U_{n,0}} \cap \overline{U_{n,1}} = \emptyset$ and

$$\forall n \in \mathbb{N} \forall s \in 2^{n+1} \forall c \in 2^{\mathbb{N}} \ \mu_{\psi_{n+1}(s) \frown c}(U_{n,s(n)}) > 1 - \epsilon_n,$$

where $\psi_n \colon 2^n \to 2^{\sum_{m<n} k_m}$ is given by $\psi_n(s) = \bigoplus_{m<n} (s(m))^{k_m}$.

Suppose that $n \in \mathbb{N}$ and we have already found k_m and $U_{m,i}$ for $i < 2$ and $m < n$. For all $s_0, s_1 \in 2^n$, the $\mathbb{E}_0$-inequivalence of the sequences $\psi_n(s_0) \frown (0)^\infty$ and $\psi_n(s_1) \frown (1)^\infty$ yields disjoint Borel sets $B_{n,s_0,s_1,0}, B_{n,s_0,s_1,1} \subseteq X$ with $\mu_{\psi_n(s_i) \frown (i)^\infty}(B_{n,s_0,s_1,i}) = 1$ for all $i < 2$. The tightness of Borel probability measures on Polish spaces (see, for example, [Kec95, Theorem 17.11]) then gives rise to compact sets $K_{n,s_0,s_1,i} \subseteq B_{n,s_0,s_1,i}$ such that $\mu_{\psi_n(s_i) \frown (i)^\infty}(K_{n,s_0,s_1,i}) > 1 - \epsilon_n / 2^n$ for all $i < 2$. The disjointness of $K_{n,s_0,s_1,0}$ and $K_{n,s_0,s_1,1}$ ensures that they are of positive distance apart, in which case they are contained in open sets $U_{n,s_0,s_1,0}$ and $U_{n,s_0,s_1,1}$ with disjoint closures. As each $K_{n,s_0,s_1,i}$ is compact, we can assume that these sets are expressible as finite unions of open sets of diameter at most δ_n.

Now define $U_{n,s_i,i} = \bigcap_{s_{1-i} \in 2^n} U_{n,s_0,s_1,i}$ and $U_{n,i} = \bigcup_{s_i \in 2^n} U_{n,s_i,i}$. The latter sets are finite unions of open sets of diameter at most δ_n, since this property is closed under finite intersections and unions. Moreover, an elementary calculation reveals that they have disjoint closures and $\mu_{\psi_n(s) \frown (i)^\infty}(U_{n,i}) > 1 - \epsilon_n$ for all $i < 2$ and $s \in 2^n$. As each of the functions $\mu \mapsto \mu(U_{n,i})$ is upper semi-continuous (see, for example, [Kec95, Corollary 17.21]), there is a natural number $k_n > 0$ such that

$$\forall i < 2 \forall s \in 2^n \forall c \in 2^{\mathbb{N}} \ \mu_{\psi_n(s) \frown (i)^{k_n} \frown c}(U_{n,i}) > 1 - \epsilon_n.$$

This completes the construction. As the sequences $(K_{n,c})_{c \in 2^{\mathbb{N}}}$ given by $K_{n,c} = \bigcap_{m \geq n} \overline{U_{m,c(m)}}$ are closed and totally bounded, they are compact (see, for example, [Kec95, Proposition 4.2]). It follows that the sequence $(K_c)_{c \in 2^{\mathbb{N}}}$ given by $K_c = \bigcup_{n \in \mathbb{N}} K_{n,c}$ is K_σ. Clearly $K_c = K_d$ whenever $c \mathrel{\mathbb{E}_0} d$, and $K_c \cap K_d = \emptyset$ whenever $\neg c \mathrel{\mathbb{E}_0} d$. Define $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\psi(x) = \bigcup_{n \in \mathbb{N}} \psi_n(x \upharpoonright n)$, and observe that

$$\forall c \in 2^{\mathbb{N}} \ \mu_{\psi(c)}(K_c) = \sup_{n \in \mathbb{N}} \mu_{\psi(c)}(K_{n,c}) \geq 1 - \inf_{n \in \mathbb{N}} \sum_{m \geq n} \epsilon_m = 1,$$

thus the function given by $\phi([c]_{\mathbb{E}_0}) = [\psi(c)]_{\mathbb{E}_0}$ is as desired. ⊠

A *homomorphism* from a subset R of $X \times X$ to a subset S of $Y \times Y$ is a function $\phi \colon X \to Y$ such that $\forall (x_1, x_2) \in R \ (\phi(x_1), \phi(x_2)) \in S$. More generally, a *homomorphism* from a pair (R_1, R_2) of subsets of X to a pair (S_1, S_2)

of subsets of Y is a function $\phi\colon X \to Y$ which is both a homomorphism from R_1 to S_1 and a homomorphism from R_2 to S_2. A *reduction* of R to S is a homomorphism from $(R, \sim R)$ to $(S, \sim S)$, and an *embedding* of R into S is an injective reduction.

A *standard Borel space* is a set X equipped with the family of Borel sets associated with a Polish topology on X. Every subset of such a space inherits the σ-algebra consisting of its intersection with each Borel subset of the original space; this restriction is again standard Borel exactly when the subset in question is Borel (see, for example, [Kec95, Corollary 13.4 and Theorem 15.1]). A function between such spaces is *Borel* if pre-images of Borel sets are Borel.

A Borel equivalence relation E on a standard Borel space is *smooth* if it is Borel reducible to equality on a standard Borel space. The Harrington-Kechris-Louveau $\mathbb{E}_0$ dichotomy ensures that this is equivalent to the inexistence of a Borel embedding of $\mathbb{E}_0$ into E (see [HKL90, Theorem 1.1]).

Following the standard abuse of language, we say that an equivalence relation is *countable* if all of its classes are countable, and *finite* if all of its classes are finite.

Suppose that E is a countable Borel equivalence relation. We say that a Borel measure μ on X is *E-ergodic* if every E-invariant Borel set is μ-conull or μ-null. The *E-saturation* of a subset Y of X is given by $[Y]_E = \{x \in X \mid \exists y \in Y \; x \; E \; y\}$. As a result of Lusin's ensures that images of Borel sets under countable-to-one Borel functions are again Borel (see, for example, [Kec95, Lemma 18.12]), it follows that E-saturations of Borel sets are Borel. We say that μ is *E-quasi-invariant* if the E-saturation of every null Borel subset of X is again null. The *push forward* of a Borel measure μ on X through a Borel function $\phi\colon X \to Y$ is the Borel measure on Y given by $(\phi_* \mu)(B) = \mu(\phi^{-1}(B))$. The Lusin-Novikov uniformization theorem for Borel subsets of the plane with countable vertical sections (see, for example, [Kec95, Theorem 18.10]) ensures that a Borel probability measure μ on X is E-quasi-invariant if and only if $\mu \sim T_* \mu$ for every Borel automorphism $T\colon X \to X$ whose graph is contained in E. We use $\mathcal{EQ}_E$ to denote the set of E-ergodic, E-quasi-invariant Borel probability measures on X. By [Dit92, Theorem 2 of Chapter 2], the set $\mathcal{EQ}_E$ is Borel whenever E is a countable Borel equivalence relation.

Recall that a subset of a standard Borel space is *analytic* if it is the image of a Borel subset of a standard Borel space under a Borel function, and *co-analytic* if its complement is analytic. Much as the previously mentioned result of Burgess-Mauldin and Souslin's perfect set theorem for analytic sets (see, for example, [Kec95, Exercise 14.13]) yield a strengthening of the special case of the latter for families of pairwise orthogonal Borel probability measures, we have the following strengthening of a special case of the $\mathbb{E}_0$ dichotomy.

Theorem 2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $A \subseteq \mathcal{EQ}_E$ is an analytic set. Then exactly one of the following holds:*

1. *The restriction of measure equivalence to A is smooth.*
2. *There is a continuous embedding $\pi \colon 2^{\mathbb{N}} \to A$ of $\mathbb{E}_0$ into measure equivalence such that the sequence $(M_C)_{C \in 2^{\mathbb{N}}/\mathbb{E}_0}$, given by $M_{[c]_{\mathbb{E}_0}} = [\pi(c)]_\sim$ for $c \in 2^{\mathbb{N}}$, has a K_σ witness to orthogonality.*

Proof. The (easy direction of) the $\mathbb{E}_0$ dichotomy yields (1) $\implies \neg$(2). To see $\neg$(1) $\implies$ (2), appeal once more to the $\mathbb{E}_0$ dichotomy to obtain a continuous embedding $\phi \colon 2^{\mathbb{N}} \to A$ of $\mathbb{E}_0$ into measure equivalence. Theorem 1 then yields a continuous embedding $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\mathbb{E}_0$ into $\mathbb{E}_0$ such that the sequence $(M_C)_{C \in 2^{\mathbb{N}}/\mathbb{E}_0}$, given by $M_{[c]_{\mathbb{E}_0}} = [(\phi \circ \psi)(c)]_\sim$ for $c \in 2^{\mathbb{N}}$, has a K_σ witness to orthogonality. Define $\pi = \phi \circ \psi$. ⊠

We next make note of a useful sufficient condition for orthogonality. As before, we say that a sequence $(\mu_i)_{i \in I}$ of Borel probability measures on X is *Borel* if it is Borel when viewed as a function from I to $P(X)$, and we say that a sequence $(B_i)_{i \in I}$ of Borel subsets of X is *Borel* if the corresponding set $\{(i, x) \in I \times X \mid x \in B_i\}$ is Borel.

Proposition 3. *Suppose that I, J, and X are Polish spaces, E is a countable Borel equivalence relation on X, μ and ν are Borel probability measures on I and J, $(\mu_i)_{i \in I}$ and $(\nu_j)_{j \in J}$ are Borel sequences of E-ergodic Borel probability measures on X which have Borel witnesses $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ to orthogonality consisting solely of E-invariant sets, and μ_i and ν_j are orthogonal for all $i \in I$ and $j \in J$. Then the measures $\mu' = \int \mu_i \; d\mu(i)$ and $\nu' = \int \nu_j \; d\nu(j)$ are orthogonal as well.*

Proof. As a result of Lusin-Souslin ensures that images of Borel sets under Borel injections are Borel (see, for example, [Kec95, Theorem 15.1]), it follows that the sets $A = \bigcup_{i \in I} A_i$ and $B = \bigcup_{j \in J} B_j$ are Borel.

We say that a sequence $(J_n)_{n \in \mathbb{N}}$ of subsets of J *separates points* if for all distinct $j, k \in J$, there exists $n \in \mathbb{N}$ such that $j \in J_n$ and $k \notin J_n$. Given any such sequence of Borel sets, the corresponding sets $B_n = \bigcup_{j \in J_n} B_j$ are also Borel, as above. Note that if $i \in I$ and $\mu_i(B) = 1$, then for each $n \in \mathbb{N}$, either $\mu_i(B_n) = 1$ or $\mu_i(B \setminus B_n) = 1$. In particular, it follows that there is a unique $j \in J$ for which $\mu_i(B_j) = 1$.

Set $I_k = \{i \in I \mid \mu_i(B) = k\}$ for $k < 2$. For each $i \in I_1$, let $\phi(i)$ denote the unique $j \in J$ for which $\mu_i(B_j) = 1$. As the graph of ϕ is Borel (see, for example, [Kec95, Theorem 17.25]), so too is ϕ (see, for example, [Kec95, Proposition 12.4]).

Fix an enumeration $(U_n)_{n \in \mathbb{N}}$ of a basis for X. For each $s \in \mathbb{N}^{\mathbb{N}}$, set $X_s = \bigcup_{n \in \mathbb{N}} U_{s(n)}$, and for each $s \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, set $X_s = \bigcap_{n \in \mathbb{N}} X_{s(n)}$.

By the Jankov-von Neumann uniformization theorem for analytic subsets of the plane (see, for example, [Kec95, Theorem 18.1]), there is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable function $\psi \colon I_1 \to (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ with the property that $X_{\psi(i)}$ is μ_i-conull and $\nu_{\phi(i)}$-null, for all $i \in I_1$. Then there is a $(\mu \restriction I_1)$-conull Borel set $C_1 \subseteq I_1$ on which ψ is Borel (see, for example, [Kec95, Exercise 17.15]).

Set $A'_i = A_i \setminus B$ for all $i \in I_0$, and $A'_i = (A_i \setminus B) \cup (A_i \cap B_{\phi(i)} \cap X_{\psi(i)})$ for all $i \in C_1$. Clearly $\mu_i(A'_i) = 1$ for all i in the set $C' = I_0 \cup C_1$, so the set $A' = \bigcup_{i \in C'} A'_i$ is μ'-conull. Observe now that if $j \in J$ and $\nu_j(A') > 0$, then, as above, there exists a unique $i \in I$ for which $\nu_j(A'_i) > 0$. Then $A'_i \cap B_j \neq \emptyset$, so $\phi(i) = j$, in which case the definition of A'_i ensures that it is ν_j-null, a contradiction. It follows that A' is ν_j-null for all $j \in J$, and therefore ν'-null. ⊠

We say that a Borel equivalence relation E on X is *hyperfinite* if there is an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations on X whose union is E. We say that E is *μ-hyperfinite* if there is a μ-conull Borel set on which E is hyperfinite, and we use $\mathcal{H}_E$ to denote the set of Borel probability measures μ on X for which E is μ-hyperfinite. Segal has shown that if E is a countable Borel equivalence relation, then the set $\mathcal{H}_E$ is Borel (see, for example, [CM14, Theorem J.10]).

We say that an E-ergodic Borel probability measure μ on X is *(E, F)-ergodic* if for every Borel homomorphism $\phi \colon X \to Y$ from E to F, there exists $y \in Y$ such that $\mu(\phi^{-1}([y]_F)) = 1$. When $F = \mathbb{E}_0$, this is also referred to as being *strongly ergodic*. As noted in [CM14, Proposition 4.10], if $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a countable union of measure-equivalence classes, then $\mathcal{EQ}_E \setminus \mathcal{H}_E$ consists solely of such Borel measures. The primary result of this paper is the following strengthening, which characterizes the family of countable Borel equivalence relations with the latter property.

Theorem 4. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *The restriction of measure equivalence to $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is smooth.*

2. *Every measure in* $\mathcal{EQ}_E \setminus \mathcal{H}_E$ *is* $(E, \mathbb{E}_0)$*-ergodic.*

Proof. To see $\neg(1) \implies \neg(2)$, appeal to Theorem 2 to obtain a continuous embedding $\pi \colon 2^{\mathbb{N}} \to \mathcal{EQ}_E \setminus \mathcal{H}_E$ of $\mathbb{E}_0$ into measure equivalence, for which there is a witness $(B_C)_{C \in 2^{\mathbb{N}}/\mathbb{E}_0}$ to the orthogonality of $(M_C)_{C \in 2^{\mathbb{N}}/\mathbb{E}_0}$ such that the sequence $(B_c)_{c \in 2^{\mathbb{N}}}$ given by $B_c = B_{[c]_{\mathbb{E}_0}}$ is Borel, where $\mu_c = \pi(c)$ and $M_{[c]_{\mathbb{E}_0}} = [\mu_c]_\sim$ for all $c \in 2^{\mathbb{N}}$. Fix an $\mathbb{E}_0$-ergodic Borel probability measure μ on $2^{\mathbb{N}}$ which is *continuous*, in the sense that every countable set is μ-null, such as the $\mathbb{N}$-fold power of the uniform probability measure on 2 (see, for example, [Kec95, Exercise 17.1]). Let ν denote the Borel probability measure on X given by $\nu(B) = \int \mu_c(B) \; d\mu(c)$.

To see that ν is E-ergodic, note that if $B \subseteq X$ is an E-invariant Borel set, then the $\mathbb{E}_0$-invariant set $A = \{c \in 2^{\mathbb{N}} \mid \mu_c(B) = 1\}$ is also Borel, in which case $\mu(A) = 0$ or $\mu(A) = 1$, so the definition of ν ensures that $\mu(A) = \nu(B)$, thus $\nu(B) = 0$ or $\nu(B) = 1$. To see that ν is E-quasi-invariant, note that if $B \subseteq X$ is Borel, then

$$\begin{aligned} \nu(B) > 0 &\iff \mu(\{c \in 2^{\mathbb{N}} \mid \mu_c(B) > 0\}) > 0 \\ &\iff \mu(\{c \in 2^{\mathbb{N}} \mid \mu_c([B]_E) > 0\}) > 0 \\ &\iff \nu([B]_E) > 0. \end{aligned}$$

To see that E is not ν-hyperfinite, note that otherwise there is a μ-positive (in fact, μ-conull) Borel set of $c \in 2^{\mathbb{N}}$ for which E is μ_c-hyperfinite, contradicting our choice of π. To see that μ is not $(E, \mathbb{E}_0)$-ergodic, appeal to the uniformization theorem for Borel subsets of the plane with countable vertical sections to obtain a Borel homomorphism $\phi \colon X \to 2^{\mathbb{N}}$ from E to $\mathbb{E}_0$ such that $c \mathrel{\mathbb{E}_0} \phi(x)$ for all $c \in 2^{\mathbb{N}}$ and $x \in B_c$, and observe that if $c, d \in 2^{\mathbb{N}}$ are $\mathbb{E}_0$-inequivalent, then $\mu_d(\phi^{-1}(c)) = 0$. As μ is continuous, it follows that $\nu(\phi^{-1}(c)) = 0$ for all $c \in 2^{\mathbb{N}}$.

To see $(1) \implies (2)$, suppose, towards a contradiction, that the restriction of measure equivalence to $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is smooth, but there exists $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$ which is not $(E, \mathbb{E}_0)$-ergodic. Fix a μ-null-to-one Borel homomorphism $\phi \colon X \to 2^{\mathbb{N}}$ from E to $\mathbb{E}_0$, as well as a Borel sequence $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X which forms a *disintegration* of μ through ϕ, in the sense that (1) $\mu_c(\phi^{-1}(c)) = 1$ for all $c \in \phi[X]$, and (2) $\mu = \int \mu_c \; d(\phi_* \mu)(c)$ (see, for example, [Kec95, Exercise 17.35]). By the uniformization theorem for Borel subsets of the plane with countable vertical sections, there are Borel functions $\phi_n \colon X \to X$ with the property that $E = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\phi_n)$. Fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers for which $\sum_{n \in \mathbb{N}} \epsilon_n = 1$, and for each $c \in 2^{\mathbb{N}}$, let μ'_c denote the weighted sum $\sum_{n \in \mathbb{N}} \epsilon_n (\phi_n)_* \mu_c$.

Lemma 5. *Suppose that $B \subseteq 2^{\mathbb{N}}$ is a $(\phi_* \mu)$-positive Borel set. Then $\mathbb{E}_0 \upharpoonright B$ is non-smooth.*

Proof. Suppose, towards a contradiction, that there is a standard Borel space Y and a Borel reduction $\psi \colon B \to Y$ of $\mathbb{E}_0 \upharpoonright B$ to equality. The countability of E ensures that ψ is countable-to-one, thus $\psi \circ \phi$ is μ-null-to-one. However, $\mu \upharpoonright \phi^{-1}(B)$ is $(E \upharpoonright \phi^{-1}(B))$-ergodic and $\psi \circ \phi$ is a homomorphism from $E \upharpoonright \phi^{-1}(B)$ to equality, so $\psi \circ \phi$ is constant on a μ-conull Borel subset of $\phi^{-1}(B)$, the desired contradiction. ⊠

In particular, it follows that if we can find a $(\phi_* \mu)$-conull Borel subset of $2^{\mathbb{N}}$ on which the function $c \mapsto \mu'_c$ is a reduction of $\mathbb{E}_0$ to the restriction of measure equivalence to $\mathcal{EQ}_E \setminus \mathcal{H}_E$, then it will follow that the latter is non-smooth, the desired contradiction.

Towards this end, note that each μ'_c is E-quasi-invariant, and the fact that ϕ is a homomorphism from E to $\mathbb{E}_0$ ensures that $\phi^{-1}([c]_{\mathbb{E}_0})$ is μ'_c-conull for all $c \in 2^{\mathbb{N}}$, thus $\mu'_c \perp \mu'_d$ whenever $\neg c \; \mathbb{E}_0 \; d$. It is therefore sufficient to establish the existence of a $(\phi_* \mu)$-conull Borel set $C \subseteq 2^{\mathbb{N}}$ with the property that for all $c \in C$, the following conditions hold:

a. The relation E is not μ'_c-hyperfinite.
b. The measure μ'_c is E-ergodic.
c. For all $d \in C \cap [c]_{\mathbb{E}_0}$, the measures μ'_c and μ'_d are equivalent.

To handle condition (a), note that by results of Segal, the set C of $c \in 2^{\mathbb{N}}$ for which E is not μ_c-hyperfinite is Borel, and therefore $(\phi_* \mu)$-conull (see, for example, [CM14, Theorems J.10 and J.13]). But for each $c \in 2^{\mathbb{N}}$, the μ_c-hyperfiniteness of E and the μ'_c-hyperfiniteness of E are equivalent (see, for example, [DJK94, Proposition 5.2]).

A function $\rho \colon E \to \mathbb{R}^+$ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$ whenever $x \; E \; y \; E \; z$. For each $c \in 2^{\mathbb{N}}$, fix a Borel cocycle $\rho'_c \colon E \to \mathbb{R}^+$ with respect to which μ'_c is *invariant*, in the sense that

$$(T_* \mu)(B) = \int_B \rho'_c(T^{-1}(x), x) \; d\mu'_c(x),$$

whenever $B \subseteq X$ is a Borel set and $T \colon X \to X$ is a Borel automorphism whose graph is contained in E. The existence of such cocycles follows, for example, from [KM04, Proposition 8.3], and a rudimentary inspection of the proof of the latter reveals that it is sufficiently uniform so as to ensure the existence of such cocycles for which the corresponding sequence $(\rho'_c)_{c \in 2^{\mathbb{N}}}$ is Borel.

For each $c \in 2^{\mathbb{N}}$, fix a sequence $(\mu'_{c,x})_{x \in X}$ of E-ergodic ρ'_c-invariant Borel probability measures on X which forms an *ergodic decomposition* of ρ'_c, in the sense that (1) $\mu'(\{x \in X \mid \mu'_{c,x} = \mu'\}) = 1$ for all E-ergodic ρ'_c-invariant Borel probability measures μ' on X, and (2) $\mu' = \int \mu'_{c,x} \, d\mu'(x)$ for all ρ'_c-invariant Borel probability measures μ' on X. Ditzen has established the existence of such sequences (see [Dit92, Theorem 6 of Chapter 2]), and a rudimentary inspection of the proof again reveals that it is sufficiently uniform so as to ensure the existence of such decompositions for which the corresponding sequence $(\mu'_{c,x})_{c \in 2^{\mathbb{N}}, x \in X}$ is Borel.

A result of Segal's ensures that for all $c \in C$, the set of $x \in X$ for which E is not $\mu'_{c,x}$-hyperfinite is μ'_c-positive (see, for example, [CM14, Proposition J.12]). For each $c \in 2^{\mathbb{N}}$, let λ'_c denote the Borel measure on $P(X)$ given by $\lambda'_c(M) = \mu'_c(\{x \in X \mid \mu'_{c,x} \in M\})$.

Note that for all Borel measures ν on X, there is at most one $\mathbb{E}_0$-class whose preimage under ϕ is ν-conull. Let M denote the set of Borel measures in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ for which there is exactly one such $\mathbb{E}_0$-class. As the graph of ϕ is Borel (see, for example, [Kec95, Proposition 12.4]), it follows that $\{(c, x) \in 2^{\mathbb{N}} \times X \mid c \ \mathbb{E}_0 \ \phi(x)\}$ is also Borel, thus so too is M. As the restriction of measure equivalence to $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is smooth, the quotient $M/{\sim}$ can be viewed as an analytic subset of a standard Borel space, in which case the uniformization theorem for analytic subsets of the plane with countable vertical sections (see, for example, [Kec95, Exercise 35.13]) yields $\sim$-invariant analytic sets $M_n \subseteq M$ and Borel functions $\psi_n \colon M \to 2^{\mathbb{N}}$ such that $M = \bigcup_{n \in \mathbb{N}} M_n$, $\phi^{-1}([\psi_n(\nu)]_{\mathbb{E}_0})$ is ν-conull, and $\psi_n(\nu)$ depends only on the measure-equivalence class of ν, for all $n \in \mathbb{N}$ and $\nu \in M_n$.

Fix enumerations $(U_n)_{n \in \mathbb{N}}$ and $(V_n)_{n \in \mathbb{N}}$ of bases for X and $P(X)$, respectively. For each $s \in \mathbb{N}^{\mathbb{N}}$, set $X_s = \bigcup_{n \in \mathbb{N}} U_{s(n)}$ and $P_s = \bigcup_{n \in \mathbb{N}} V_{s(n)}$. For each $s \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, set $X_s = \bigcap_{n \in \mathbb{N}} X_{s(n)}$ and $P_s = \bigcap_{n \in \mathbb{N}} P_{s(n)}$.

Suppose now, towards a contradiction, that no $(\phi_* \mu)$-conull Borel subset of C satisfies conditions (b) and (c), and let R denote the set of quadruples (c, c', d, d') of pairwise $\mathbb{E}_0$-related points of C for which there exist $i, j \in \mathbb{N}$ and $t, u \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ such that $\lambda'_c(\psi_i^{-1}(c') \cap M_i \cap (P_t \setminus P_u))$ and $\lambda'_d(\psi_j^{-1}(d') \cap M_j \cap (P_u \setminus P_t))$ are non-zero.

Lemma 6. *The projection of R onto the leftmost coordinate contains a $(\phi_* \mu)$-positive Borel set.*

Proof. Define $B = \{c \in C \mid \mu'_c \text{ is not } E\text{-ergodic}\}$. If $c \in B$, then there exist disjoint E-invariant μ'_c-positive Borel subsets Y and Z of X. The definition of ergodic decomposition then ensures that the set P of measures $\nu \in M$ with respect to which Y is conull is λ'_c-positive, as is the set Q of measures $\nu \in M$ with respect to which Z is conull. And clearly these sets are themselves disjoint. As Borel probability measures on Polish spaces are regular (see, for example, [Kec95, Theorem 17.10]), there exist $t, u \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ such that both $P_t \mathbin{\triangle} P$ and $P_u \mathbin{\triangle} Q$ are λ'_c-null, in which case both $M \cap (P_t \setminus P_u)$ and $M \cap (P_u \setminus P_t)$ are λ'_c-positive. Fix $c', d' \in [c]_{\mathbb{E}_0}$ and $i, j \in \mathbb{N}$ such that $\lambda'_c(\psi_i^{-1}(c') \cap M_i \cap (P_t \setminus P_u))$ and $\lambda'_c(\psi_j^{-1}(d') \cap M_j \cap (P_u \setminus P_t))$ are non-zero. Then $(c, c', c, d') \in R$, so c is in the projection of R onto the leftmost coordinate. As [Dit92, Theorem 2 of Chapter 2] ensures that B is Borel, we can therefore assume that B is $(\phi_*\mu)$-null.

Define $A = \{c \in C \setminus B \mid \exists d \in (C \setminus B) \cap [c]_{\mathbb{E}_0}\ \mu'_c \not\sim \mu'_d\}$. If $c \in A$, then there exists $d \in (C \setminus B) \cap [c]_{\mathbb{E}_0}$ for which there are disjoint E-invariant Borel subsets Y and Z of X which are μ'_c-conull and μ'_d-conull, respectively. The definition of ergodic decomposition then ensures that the set P of measures $\nu \in M$ with respect to which Y is conull is λ'_c-positive, and the set Q of measures $\nu \in M$ with respect to which Z is conull is λ'_d-positive. And clearly these sets are disjoint. Appealing again to the regularity of Borel probability measures on Polish spaces, we obtain sequences $t, u \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ for which $P_t \mathbin{\triangle} P$ is λ'_c-null and $P_u \mathbin{\triangle} Q$ is λ'_d-null, in which case $M \cap (P_t \setminus P_u)$ is λ'_c-positive, and $M \cap (P_u \setminus P_t)$ is λ'_d-positive. Fix $c', d' \in [c]_{\mathbb{E}_0}$ and $i, j \in \mathbb{N}$ such that $\lambda'_c(\psi_i^{-1}(c') \cap M_i \cap (P_t \setminus P_u))$ and $\lambda'_d(\psi_j^{-1}(d') \cap M_j \cap (P_u \setminus P_t))$ are non-zero. Then $(c, c', d, d') \in R$, so c is in the projection of R onto the leftmost coordinate. As countable-to-one images of Borel sets under Borel functions are Borel, it follows that A is Borel. As $C \setminus (A \cup B)$ satisfies conditions (b) and (c), it is not $(\phi_*\mu)$-conull, and it follows that A is $(\phi_*\mu)$-positive. ⊠

A result of Kondô-Tugué ensures that R is analytic (see, for example, [Kec95, Theorem 29.26]). As a result of Lusin's ensures that analytic sets are universally measurable (see, for example, [Kec95, Theorem 29.7]), it follows that there is an analytic set $S \subseteq R$, whose projection onto the leftmost coordinate still contains a $(\phi_*\mu)$-positive Borel set, but having the additional property that for all distinct quadruples $(c, c', d, d'), (e, e', f, f') \in S$, the sets $\{c, c', d, d'\}$ and $\{e, e', f, f'\}$ are disjoint (this can be seen, for example, as a straightforward consequence of [CM14, Propositions 3 and 7]). Moreover, we can also assume that S is Borel, in which case the fact that $\phi_*\mu$ is continuous allows us to assume that the projection of S onto both the leftmost and penultimate coordinates are $(\phi_*\mu)$-positive.

The uniformization theorem for analytic subsets of the plane then yields a $\sigma(\mathbf{\Sigma}^1_1)$-measurable function associating with every quadruple $s = (c, c', d, d')$ in S a quadruple $(i_s, j_s, t_s, u_s) \in \mathbb{N} \times \mathbb{N} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ such that $\lambda'_c(\psi_{i_s}^{-1}(c') \cap M_{i_s} \cap (P_{t_s} \setminus P_{u_s}))$ and $\lambda'_d(\psi_{j_s}^{-1}(d') \cap M_{j_s} \cap (P_{u_s} \setminus P_{t_s}))$ are non-zero. By throwing out a Borel subset of S whose projection onto each coordinate is $(\phi_* \mu)$-null, we can assume that this function is in fact Borel.

For all quadruples $s = (c, c', d, d')$ in S, set

$$\mu''_c = \int_{\psi_{i_s}^{-1}(c') \cap M_{i_s} \cap (P_{t_s} \setminus P_{u_s})} \nu \; d\mu'_c(\nu)$$

and

$$\mu''_d = \int_{\psi_{j_s}^{-1}(d') \cap M_{j_s} \cap (P_{u_s} \setminus P_{t_s})} \nu \; d\mu'_d(\nu).$$

Proposition 3 ensures that these two measures are orthogonal, as are $\mu''_c + \mu''_d$ and $\mu''_e + \mu''_f$ whenever $(c, c', d, d'), (e, e', f, f') \in S$ are distinct.

By the uniformization theorem for analytic subsets of the plane, there is a $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable function associating with every quadruple $s = (c, c', d, d')$ in S a pair $(v_s, w_s) \in (\mathbb{N}^\mathbb{N})^\mathbb{N} \times (\mathbb{N}^\mathbb{N})^\mathbb{N}$ with the property that $X_{v_s} \setminus X_{w_s}$ is μ''_c-conull, $X_{w_s} \setminus X_{v_s}$ is μ''_d-conull, and $X_{v_s} \cup X_{w_s}$ is $(\mu''_e + \mu''_f)$-null for all $(e, e', f, f') \in S \setminus \{s\}$ with $c \; \mathbb{E}_0 \; e$. By throwing out a Borel subset of S whose projection onto each coordinate is $(\phi_* \mu)$-null, we can assume that this function is also Borel.

As images of Borel sets under countable-to-one Borel functions are Borel, it follows that the sets

$$\bigcup \{\phi^{-1}([c]_{\mathbb{E}_0}) \cap X_{v_s} \mid s = (c, c', d, d') \text{ is in } S\}$$

and

$$\bigcup \{\phi^{-1}([c]_{\mathbb{E}_0}) \cap X_{w_s} \mid s = (c, c', d, d') \text{ is in } S\}$$

are Borel. Moreover, they are both μ-positive, but their E-saturations have μ-null intersection, contradicting the E-ergodicity of μ. ⊠

Remark 7. As there is a homeomorphism $\phi \colon 2^\mathbb{N} \to 2^\mathbb{N} \times 2^\mathbb{N}$ sending $\mathbb{E}_0$ to $\mathbb{E}_0 \times \mathbb{E}_0$, it follows that if there is a measure $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$ which is not $(E, \mathbb{E}_0)$-ergodic, then the restriction of measure equivalence to the family of all such measures is non-smooth.

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