Borel homomorphisms of smooth σ-ideals

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Abstract

Given a countable Borel equivalence relation E on a Polish space, let $\mathcal{I}_E$ denote the σ-ideal generated by the Borel partial transversals of E. We show that there is a Borel homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$ if and only if there is a smooth-to-one Borel homomorphism from a finite index Borel subequivalence relation of E to F. As a corollary, we see that $\mathcal{I}_E$ is homogeneous in the sense of Zapletal (2007, Forcing Idealized, Preprint) if and only if E is hyperfinite. Using this, we prove that all $\mathbf{\Sigma^1_2}$ sets and $\mathbf{\Sigma^1_1}$ quasi-orders are Borel on Borel reducible to the quasi-order of Borel homomorphism on the class of inhomogeneous $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$ σ-ideals.

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1 Introduction

Suppose that E and F are countable Borel equivalence relations on Polish spaces X and Y. A *reduction* of E to F is a map $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ (x_1 E x_2 \iff \pi(x_1) F \pi(x_2)),$$

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and E is *Borel reducible* to F, or $E \leq_B F$, if there is a Borel reduction of E to F. The study of Borel reducibility plays a central role in the descriptive set-theoretic study of classification problems (see Jackson-Kechris-Louveau [8]).

A *partial transversal* of E is a set $B \subseteq X$ such that

$$\forall x_1, x_2 \in B \ (x_1 E x_2 \Rightarrow x_1 = x_2).$$

Associated with E is the σ-ideal $\mathcal{I}_E$ on X generated by the Borel partial transversals of E. Our primary goal here is to understand the extent to which descriptive set-theoretic properties of E are encoded in $\mathcal{I}_E$.

Several robust classes of countable Borel equivalence relations were isolated early on in the development of the subject, and these classes will be important for our work here. Recall that E is *smooth* if $X \in \mathcal{I}_E$, E is *hyperfinite* if there is an increasing sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ of finite Borel equivalence relations such that $E = \bigcup_{n \in \mathbb{N}} E_n$, and E is *treeable* if there is an acyclic Borel graph $\mathcal{G}$ on X whose connected components coincide with the equivalence classes of E.

Suppose that $\mathcal{E}$ is a family of countable Borel equivalence relations on standard Borel spaces. We say that an equivalence relation E' is of *finite index* below E if $E' \subseteq E$ and every E-class is the union of finitely many E'-classes, and we say that E is *almost* $\mathcal{E}$ if some equivalence relation in $\mathcal{E}$ is of finite index below E. Given a measure μ on X, we say that E is μ-$\mathcal{E}$ if there is a μ-conull Borel set $C \subseteq X$ such that $E|C \in \mathcal{E}$. We say that E is *measure* $\mathcal{E}$ if E is μ-$\mathcal{E}$ for every Borel probability measure μ on X.

Suppose that $\mathcal{I}$ and $\mathcal{J}$ are σ-ideals on X and Y. A *homomorphism* from $\mathcal{I}$ to $\mathcal{J}$ is a map $\pi : X \to Y$ such that

$$\forall B \in \mathcal{J} \ (\pi^{-1}(B) \in \mathcal{I}).$$

It is easy to see that if $\pi : X \to Y$ is a Borel reduction of E to F, then π is also a Borel homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$. While the converse is clearly false, here we prove a version of the converse for a natural weakening of Borel reducibility.

A *homomorphism* from E to F is a map $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ (x_1 E x_2 \Rightarrow \pi(x_1) F \pi(x_2)).$$

It makes little sense to study the existence of Borel homomorphisms between equivalence relations, as constant functions are necessarily homomorphisms. To avoid such degeneracies, we restrict our attention to homomorphisms which do not collapse a large portion of the complexity of E into a single point of Y. We say that a function $\pi : X \to Y$ is *$\mathcal{E}$-to-one* if $\forall y \in Y \ (E|\pi^{-1}(y) \in \mathcal{E})$.

In §2, we show that the notion of smooth-to-one Borel homomorphism is robust, in the sense that it preserves the key classes of smooth, hyperfinite, and treeable equivalence relations. We show also that the class of measure hyperfinite equivalence relations is closed under measure hyperfinite-to-one Borel homomorphism.

We say that $\pi : X \to Y$ is an *almost homomorphism* if the image of each E-class under π is contained in the union of finitely many F-classes, or equivalently, if the equivalence relation E' on X given by

$$x_1 E' x_2 \iff (x_1 E x_2 \text{ and } \pi(x_1) F \pi(x_2))$$

is of finite index below E. In §3, we outline the straightforward modifications necessary to extend the measure-theoretic rigidity arguments employed by Hjorth-Kechris [6] to produce countable Borel equivalence relations which are incomparable with respect to any quasi-order that lies between almost measure treeable-to-one Borel almost homomorphism and Borel reducibility.

With these preliminaries involving $\mathcal{E}$-to-one Borel homomorphisms out of the way, in §4 we prove our main result, which characterizes the circumstances under which there is a Borel homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$:

Theorem 1 *Suppose that E and F are countable Borel equivalence relations on Polish spaces. Then the following are equivalent:*

(1) There is a Borel homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$;
(2) There is a smooth-to-one Borel almost homomorphism from E to F.

Our proof uses the usual sort of Glimm-Effros style technique. The corresponding splitting lemma relies upon the elementary observation (which is explored in greater detail in Caicedo-Clemens-Conley-Miller [2]) that if $\mathcal{F}$ is a non-empty family of finite subsets of a set X such that $\forall S, T \in \mathcal{F}\ (S \cap T \neq \emptyset)$, then a non-empty, finite subset of X is definable from $\mathcal{F}$.

The *restriction* of $\mathcal{I}$ to a Borel set $B \subseteq X$ is given by $\mathcal{I}|B = \{A \cap B : A \in \mathcal{I}\}$. Following Zapletal [17], we say that $\mathcal{I}$ is *homogeneous* if for every Borel set $B \notin \mathcal{I}$, there is a Borel homomorphism from $\mathcal{I}$ to $\mathcal{I}|B$. In §5, we characterize the circumstances under which $\mathcal{I}_E$ is homogeneous:

Theorem 2 *Suppose that E is a countable Borel equivalence relation on a Polish space. Then E is hyperfinite $\iff$ $\mathcal{I}_E$ is homogeneous.*

It follows that if $\mathcal{I}_E$ is not homogeneous, then X is not the union of countably many Borel sets on which $\mathcal{I}_E$ is homogeneous. Modulo a positive answer to the long-standing open question of whether every measure hyperfinite equivalence relation is hyperfinite, we in fact obtain that if $\mathcal{I}_E$ is not homogeneous, then there is no hyperfinite Borel partition of X into E-invariant sets on which $\mathcal{I}_E$ is homogeneous.

Zapletal [17] has asked whether there is a natural example of an inhomogeneous σ-ideal whose corresponding forcing is proper. Now, the argument behind Theorem 4.7.3 of Zapletal [17] easily generalizes to show that if E is a countable Borel equivalence relation, then the forcing associated with $\mathcal{I}_E$ is proper. As a consequence, it follows that $\mathcal{I}_E$ is of the desired form if and only if E is not hyperfinite. Unfortunately, Theorem 1.1 of Harrington-Kechris-Louveau [7] easily implies that there is a dense subset of the forcing associated with $\mathcal{I}_E$ on which it is homogeneous. In particular, the question of whether there is a natural example of a nowhere homogeneous σ-ideal whose corresponding forcing is proper remains open.

Recall that a σ-ideal $\mathcal{I}$ on X is *$\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$* if for every Polish space W and $\mathbf{\Sigma}^1_1$ set $R \subseteq W \times X$, the set $\{w \in W : R_w \in \mathcal{I}\}$ is $\mathbf{\Pi}^1_1$. Theorem 1.1 of Harrington-Kechris-Louveau [7] easily implies that the σ-ideals of the form $\mathcal{I}_E$ are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Let $\preceq_B$ denote the quasi-order of Borel homomorphism on such σ-ideals. A *reduction* of a quasi-order $\preceq$ on X to $\preceq_B$ is a function $x \mapsto \mathcal{I}_x$ such that

$$\forall x_1, x_2 \in X \ (x_1 \preceq x_2 \iff \mathcal{I}_{x_1} \preceq_B \mathcal{I}_{x_2}).$$

A *reduction* of a set $B \subseteq X$ to $\preceq_B$ is a function $x \mapsto (\mathcal{I}_x, \mathcal{J}_x)$ such that

$$\forall x \in X \ (x \in B \iff \mathcal{I}_x \preceq_B \mathcal{J}_x).$$

We say that an assignment $x \mapsto \mathcal{I}_x$ of σ-ideals on a Polish space Z is *Borel on Borel* if for every Borel set $B \subseteq X \times Z$, the set $\{x \in X : B_x \in \mathcal{I}_x\}$ is Borel. We say that an assignment $x \mapsto (\mathcal{I}_x, \mathcal{J}_x)$ is *Borel on Borel* if the assignments $x \mapsto \mathcal{I}_x$ and $x \mapsto \mathcal{J}_x$ are both Borel on Borel. Finally, we give lower bounds on the complexity of Borel homomorphism:

Theorem 3 *Every $\mathbf{\Sigma}^1_1$ quasi-order on a Polish space is Borel on Borel reducible to $\preceq_B$. Every $\mathbf{\Sigma}^1_2$ subset of a Polish space is Borel on Borel reducible to $\preceq_B$.*

2 Smooth-to-one homomorphisms

We say that a map $\pi : X \to Y$ is *locally injective* (with respect to E) if

$$\forall x_1 E x_2 \ (\pi(x_1) = \pi(x_2) \Rightarrow x_1 = x_2),$$

and we say that π is *essentially locally injective* if there is a cover $\langle B_n \rangle_{n \in \mathbb{N}}$ of X by Borel sets such that $\forall n \in \mathbb{N}$ ($\pi | B_n$ is locally injective).

Theorem 2.1 *Suppose that X and Y are Polish spaces, E is a countable Borel equivalence relation on X, and $\pi : X \to Y$ is Borel. Then the following are equivalent:*

(1) π is essentially locally injective;
(2) π is smooth-to-one.

Proof. To see (1) $\Rightarrow$ (2), suppose that π is essentially locally injective, and fix a cover $\langle B_n \rangle_{n \in \mathbb{N}}$ of X by Borel sets such that each of the restrictions $\pi | B_n$ is locally injective. Then for each $y \in Y$ and $n \in \mathbb{N}$, the set $[\pi | B_n]^{-1}(y)$ is a partial transversal of E, so the restriction of E to the set $\pi^{-1}(y) = \bigcup_{n \in \mathbb{N}} [\pi | B_n]^{-1}(y)$ is smooth, thus π is smooth-to-one.

To see (2) $\Rightarrow$ (1), suppose that $\pi : X \to Y$ is smooth-to-one, and define an equivalence relation $F \subseteq E$ by setting

$$x_1 F x_2 \iff (x_1 E x_2 \text{ and } \pi(x_1) = \pi(x_2)).$$

Lemma 2.2 *F is smooth.*

Proof. Suppose, towards a contradiction, that F is non-smooth. By Theorem 1.1 of Harrington-Kechris-Louveau [7], there is a continuous embedding $\phi : 2^{\mathbb{N}} \to X$ of E_0 into F. The generic ergodicity of E_0 therefore ensures that there exists $y \in Y$ such that $(\pi \circ \phi)^{-1}(y)$ is comeager, so $E_0 | (\pi \circ \phi)^{-1}(y)$ is non-smooth, thus $E | \pi^{-1}(y)$ is non-smooth, the desired contradiction. □

By Lemma 2.2, there is a sequence $\langle B_n \rangle_{n \in \mathbb{N}}$ of Borel partial transversals of F which covers X. As it is clear that each of the restrictions $\pi | B_n$ is locally injective, it follows that π is essentially locally injective. □

As a consequence, we can now show that various classes of equivalence relations which are defined by structurability constraints are preserved under smooth-to-one Borel homomorphisms:

Theorem 2.3 *Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, and there is a smooth-to-one Borel homomorphism from E to F.*

(1) If F is smooth, then E is smooth.
(2) If F is hyperfinite, then E is hyperfinite.
(3) If F is treeable, then E is treeable.

Proof. Fix a smooth-to-one Borel homomorphism $\pi : X \to Y$ from E to F. By Theorem 2.1, there is a cover $\langle A_n \rangle_{n \in \mathbb{N}}$ of X by Borel sets on which π is locally injective. For each $x \in X$, let $n(x)$ denote the least natural number n such that $A_n \cap [x]_E \neq \emptyset$. Then the set $A = \{x \in X : x \in A_{n(x)}\}$ is an E-complete section on which π is locally injective.

If F is smooth, then there is a cover of Y by countably many Borel partial transversals B_n of F. Then the sets of the form $A_m \cap \pi^{-1}(B_n)$, for $m, n \in \mathbb{N}$, are Borel partial transversals of E which cover X, thus E is smooth.

If F is hyperfinite, then there is an increasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ of finite Borel equivalence relations on Y whose union is F. Define E_n on A by

$$x_1 E_n x_2 \iff (x_1 E x_2 \text{ and } \pi(x_1) F_n \pi(x_2)).$$

Then $\langle E_n \rangle_{n \in \mathbb{N}}$ is an increasing sequence of finite Borel equivalence relations on A whose union is $E|A$, so $E|A$ is hyperfinite, thus E is hyperfinite, by Proposition 1.3 of Jackson-Kechris-Louveau [8].

If F is treeable, then the idea behind the proof of part (ii) of Proposition 3.3 of Jackson-Kechris-Louveau [8] adapts in a straightforward manner to show that $E|A$ treeable, and part (iv) of Proposition 3.3 of Jackson-Kechris-Louveau [8] then implies that E is treeable. $\square$

Remark 2.4 Suppose that E' is of finite index over E. It is easy to see that if E is smooth, then so too is E'. Similarly, if E is hyperfinite, then so too is E', by Proposition 1.3 of Jackson-Kechris-Louveau [8]. As a consequence, parts (1) and (2) of Theorem 2.3 require only the existence of a smooth-to-one Borel almost homomorphism.

Before going further, we need first a basic fact concerning measure hyperfinite equivalence relations. Recall that a measure μ is E-*ergodic* if every E-invariant Borel set is μ-null or μ-conull.

Proposition 2.5 *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X which is μ-hyperfinite for every E-ergodic Borel probability measure μ on X. Then E is measure hyperfinite.*

Proof. Let $P(X)$ denote the standard Borel space of Borel probability measures on X. By arguments of Segal [16] (see also §10 of Kechris-Miller [10]), there is an increasing sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ of Borel subsets of $P(X) \times X \times X$ such that:

(1) $\forall n \in \mathbb{N}\, \forall \mu \in P(X)$ ($(E_n)_\mu$ is a finite subequivalence relation of E);
(2) $\forall \mu \in P(X)$ (E is μ-hyperfinite $\Rightarrow \mu(\{x \in X : [x]_E = \bigcup_{n \in \mathbb{N}} [x]_{E_n}\}) = 1$).

Suppose now that μ is a Borel probability measure on X. By Theorem 3.2 of Louveau-Mokobodzki [12], there is a Borel function $x \mapsto \mu_x$ such that:

(a) $\forall x \in X$ (μ_x is E-ergodic);
(b) $\forall x E y$ ($\mu_x = \mu_y$);
(c) $\forall x \in X$ ($\mu_x(\{y \in X : \mu_x = \mu_y\}) = 1$);
(d) $\mu = \int \mu_x \, d\mu(x)$.

Define equivalence relations F_n on X by setting

$$xF_ny \iff x(E_n)_{\mu_x}y,$$

and observe that

$$\mu(\{x \in X : [x]_E = \bigcup_{n\in\mathbb{N}} [x]_{F_n}\}) = \int \mu_y(\{x \in X : [x]_E = \bigcup_{n\in\mathbb{N}} [x]_{(E_n)_{\mu_y}}\})\ d\mu(y).$$

As our assumption on E ensures that the latter quantity has value 1, this implies that E is μ-hyperfinite. As μ was an arbitrary Borel probability measure on X, it follows that E is measure hyperfinite. □

As a corollary, we see that the equivalence relations determined by measure hyperfinite-to-one Borel functions are themselves measure hyperfinite:

Proposition 2.6 *Suppose that X and Y are Polish spaces, E is a countable Borel equivalence relation on X, and $\pi : X \to Y$ is a measure hyperfinite-to-one Borel function. Then the equivalence relation $F \subseteq E$ given by*

$$xFy \iff (xEy \textit{ and } \pi(x) = \pi(y))$$

is measure hyperfinite.

Proof. Suppose, towards a contradiction, that F is not measure hyperfinite. By Proposition 2.5, there is an F-ergodic Borel probability measure μ on X such that F is not μ-hyperfinite. Fix $y \in Y$ such that $\mu(\pi^{-1}(y)) = 1$, and observe that $F|\pi^{-1}(y)$ is not measure hyperfinite, the desired contradiction. □

Finally, we are ready to show that the measure hyperfinite equivalence relations are closed under measure hyperfinite-to-one Borel homomorphism:

Theorem 2.7 *Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, and there is a measure hyperfinite-to-one Borel almost homomorphism from E to F. If F is measure hyperfinite, then E is measure hyperfinite.*

Proof. By Proposition 1.3 of Jackson-Kechris-Louveau [8], we can assume that there is a measure hyperfinite-to-one Borel homomorphism $\pi : X \to Y$ from E to F. Define $E' \subseteq E$ by

$$xE'y \iff (xEy \text{ and } \pi(x) = \pi(y)).$$

Proposition 2.6 ensures that E' is measure hyperfinite. Suppose now that μ is a Borel probability measure on X, let $\nu = \pi_*\mu$, and fix a Borel set $C \subseteq Y$ and an increasing sequence $\langle F_n \rangle_{n\in\mathbb{N}}$ of finite Borel equivalence relations on C such that $\nu(C) = 1$ and $F|C = \bigcup_{n\in\mathbb{N}} F_n$. For each $n \in \mathbb{N}$, define E_n on $\pi^{-1}(C)$ by

$$xE_ny \iff (xEy \text{ and } \pi(x)F_n\pi(y)).$$

Then E_n is of finite index over E', thus Proposition 1.3 of Jackson-Kechris-Louveau [8] ensures that E_n is μ-hyperfinite. By a well known theorem of Dye [4] and Krieger [11] (see also Theorem 6.11 of Kechris-Miller [10]), it now follows that $E|\pi^{-1}(C) = \bigcup_{n\in\mathbb{N}} E_n$ is μ-hyperfinite. As μ was an arbitrary Borel probability measure on X, it follows that E is measure hyperfinite. $\square$

3 Complexity of almost homomorphism

In this section, we give several straightforward strengthenings of results of Adams-Kechris [1] and Hjorth-Kechris [6]. We say that μ is (E, F)-*ergodic* if for every Borel homomorphism $\pi : X \to Y$ from E to F, there exists $y \in Y$ such that $\mu(\pi^{-1}([y]_F)) = 1$. We say that μ is *weakly* (E, F)-*ergodic* if for every Borel homomorphism $\pi : X \to Y$ from E to F, there exists $y \in Y$ such that $\mu(\pi^{-1}(y)) > 0$. Note that if F has at least two equivalence classes, then μ is (E, F)-ergodic if and only if μ is E-ergodic and weakly (E, F)-ergodic.

Recall that E_0 is the equivalence relation on $2^{\mathbb{N}}$ given by

$$xE_0y \iff \exists n \in \mathbb{N}\, \forall m \geq n\ (x(m) = y(m)).$$

Theorem 7.1 of Dougherty-Jackson-Kechris [3] ensures that an equivalence relation is hyperfinite if and only if it is Borel reducible to E_0. It follows that if μ is (E, E_0)-ergodic, then μ is (E, F)-ergodic, for every measure hyperfinite equivalence relation F. It then follows from a well known result of Ornstein-Weiss [14] (see also Theorem 10.2 of Kechris-Miller [10]) that μ is (E, F)-ergodic, for every countable Borel equivalence relation F which is generated by a Borel action of an amenable group on a Polish space.

For each set $S \subseteq \mathrm{PRIMES}$, put $\Gamma_S = \mathbb{Z} \times (*_{p\in S}\ \mathbb{Z}/p\mathbb{Z})$ and let X_S denote the free part of the action of Γ_S on 2^{Γ_S} via the shift. Let μ_S denote the $(1/2, 1/2)$ product measure on X_S, let Γ_S act on X_S by the shift, and let E_S denote the associated orbit equivalence relation. If $|S| \geq 2$, then Γ_S is not amenable, thus μ_S is (E_S, E_0)-ergodic (see, for example, Theorem A4.1 of Hjorth-Kechris [6]).

Theorem 3.1 *Suppose that* $S \subseteq \mathrm{PRIMES}$ *is of cardinality at least* 2*,* E *is a Borel equivalence relation on* X_S *of finite index below* E_S*,* $T \subseteq \mathrm{PRIMES}$ *does not contain* S*, and* Γ_T *acts freely on a Polish space* X *by Borel automorphisms. Then* μ_S *is weakly* $(E, E^X_{\Gamma_T})$*-ergodic.*

Proof. Suppose that $\pi : X_S \to X$ is a Borel homomorphism from E to $E^X_{\Gamma_T}$.

The E_S-ergodicity of μ_S ensures that by throwing out a μ_S-null, E_S-invariant Borel subset of X_S, we can assume that there exists $n \in \mathbb{Z}^+$ such that every equivalence class of E_S is the disjoint union of n equivalence classes of E. By the Lusin-Novikov uniformization theorem (see, for example, Theorem 18.10 of Kechris [9]), there are Borel functions $\pi_i : X_S \to X$ such that

$$\forall x \in X_S \, \forall y \in [x]_{E_S} \, \exists 1 \leq i \leq n \; (\pi(y) E^X_{\Gamma_T} \pi_i(x)).$$

We use S_n to denote the symmetric group on $\{1, \ldots, n\}$. Let $\alpha : \Gamma_S \times X_S \to S_n \times (\Gamma_T)^n$ denote the unique function such that

$$\forall 1 \leq i \leq n \; (\pi_i(\gamma \cdot x) = \alpha_i(\gamma, x) \cdot \pi_{[\alpha_0(\gamma,x)](i)}(x)),$$

for all $\gamma \in \Gamma_S$ and $x \in X_S$, where $\alpha(\gamma, x) = \langle \alpha_0(\gamma, x), \alpha_1(\gamma, x), \ldots, \alpha_n(\gamma, x) \rangle$. For each $1 \leq i \leq n$, fix $\alpha_{i0} : \Gamma_S \times X_S \to \mathbb{Z}$ and $\alpha_{i1} : \Gamma_S \times X_S \to *_{p \in T} \mathbb{Z}/p\mathbb{Z}$ such that $\alpha_i = \langle \alpha_{i0}, \alpha_{i1} \rangle$. It is clear that α is a Borel cocycle, thus so too are the functions $\beta_i : \Gamma_S \times X_S \to *_{p \in T} \mathbb{Z}/p\mathbb{Z}$ given by $\beta_i = \alpha_{i1}$, for $1 \leq i \leq n$.

Lemma 3.2 *Suppose that $1 \leq i \leq n$. Then there is an amenable group $\Delta_i \subseteq *_{p \in T} \mathbb{Z}/p\mathbb{Z}$ such that off of a μ_S-null, E_S-invariant Borel set, there is a Borel cocycle $\beta'_i \sim \beta_i$ and $\beta'_i(\Gamma_S \times X_S) \subseteq \Delta_i$.*

Proof. By Theorem 2.2 of Hjorth-Kechris [6], we can assume that there is a finite group $\Delta \leq *_{p \in T} \mathbb{Z}/p\mathbb{Z}$ and a Borel cocycle $\beta \sim \beta_i$ such that $\beta(\mathbb{Z} \times X_S) \subseteq \Delta$ and $\forall \gamma \in \Gamma_S$ ($x \mapsto \Delta\beta(\gamma, x)\Delta$ is constant). As in the proof of Theorem 3.1 of Hjorth-Kechris [6], it follows from Theorem 11.57 of Rotman [15] that there exists $\delta \in *_{p \in T} \mathbb{Z}/p\mathbb{Z}$ and $p \in T$ such that $\delta\Delta\delta^{-1} \subseteq \mathbb{Z}/p\mathbb{Z}$. Set $\beta'_i = \delta\beta\delta^{-1}$ and $\Delta_i = \mathbb{Z}/p\mathbb{Z}$, noting that $\beta'_i(\mathbb{Z} \times X_S) \subseteq \Delta_i$. As in the proof of Theorem 3.1 of Hjorth-Kechris [6], it now follows that $\beta'_i(\Gamma_S \times X_S) \subseteq \Delta_i$. □

Fix Borel functions $\lambda_i : X_S \to *_{p \in T} \mathbb{Z}/p\mathbb{Z}$ such that

$$\beta_i(\gamma, x) = \lambda_i(\gamma \cdot x)\beta'_i(\gamma, x)\lambda_i(x)^{-1},$$

for all $1 \leq i \leq n$, $\gamma \in \Gamma_S$, and $x \in X_S$. Define $\phi : X_S \to X$ by

$$\phi(x) = (1_{S_n}, \lambda_1(x), \ldots, \lambda_n(x))^{-1} \cdot \pi(x),$$

and observe that ϕ is a homomorphism of E_S into the equivalence relation generated by the amenable group $S_n \times \Delta_1 \times \cdots \times \Delta_n$. As μ_S is (E_S, E_0)-ergodic, there exists $x \in X$ such that $\mu_S(\phi^{-1}([x]_E)) = 1$, and it follows that there exists $y \in [x]_E$ such that $\mu_S(\pi^{-1}(y)) > 0$. □

Corollary 3.3 *If $S, T \subseteq$ PRIMES, $|S| \geq 2$, and $S \not\subseteq T$, then there is no μ_S-almost treeable-to-one Borel almost homomorphism from E_S to E_T.*

Proof. Suppose that $\pi : X_S \to X_T$ is a Borel almost homomorphism from E_S

to E_T. Define $E \subseteq E_S$ by

$$xEy \iff (xE_S y \text{ and } \pi(x)E_T\pi(y)).$$

Then E is of finite index below E_S, so Theorem 3.1 implies that there exists $x \in X_T$ such that $\mu_S(\pi^{-1}(x)) > 0$. Then $E|\pi^{-1}(x)$ is not μ_S-almost treeable by Proposition 3.3 and Theorem 3.29 of Jackson-Kechris-Louveau [8], thus π is not μ_S-almost treeable-to-one. □

We say that a function $\pi : X \to \mathcal{P}(Y)$ is *Borel* if the set

$$\text{graph}(\pi) = \{(x, y) \in X \times Y : y \in \pi(x)\}$$

is Borel. An *embedding* of E into F is an injective reduction of E to F.

Theorem 3.4 *Suppose that $\leq$ is a quasi-order on equivalence relations which sits between measure almost treeable-to-one Borel almost homomorphism and Borel embeddability.*

(1) Every $\mathbf{\Sigma^1_2}$ subset of a Polish space is Borel reducible to $\leq$.
(2) Every $\mathbf{\Sigma^1_1}$ quasi-order on a Polish space is Borel reducible to $\leq$.

Proof. The proof of (1) is just as in the proof of Theorem 5.1 of Adams-Kechris [1], using Corollary 3.3 in place of Theorem 4.2 of Adams-Kechris [1]. To obtain (2), we will employ a modification of the idea behind the proof of Theorem 4.1 of Adams-Kechris [1] in the spirit of the proof of Theorem 3 of Gao [5]. By Theorem 5.1 of Louveau-Rosendal [13], there is a complete $\mathbf{\Sigma^1_1}$ quasi-order $\preceq$ on a Polish space X which is induced by an action of a Polish monoid G. It is enough to show that $\preceq$ is Borel reducible to $\leq$.

Fix a Borel assignment $x \mapsto S_x$ of sets of primes to points of X such that

$$\forall x \in X \ (|S_x| \geq 2) \text{ and } \forall x, y \in X \ (x \neq y \Rightarrow S_x \not\subseteq S_y).$$

Set $R = \{(g, y, z) : g \in G \text{ and } y \in X \text{ and } z \in X_{S_y}\}$, and for each $x \in X$, let E_x denote the equivalence relation on R obtained by putting $(g_1, y_1, z_1)E_x(g_2, y_2, z_2)$ if either $(g_1, y_1, z_1) = (g_2, y_2, z_2)$ or $g_1 = g_2$, $y_1 = y_2$, $x = g_1 \cdot y_1$, and $z_1 E_{S_{y_1}} z_2$.

Observe that if $x_1 \preceq x_2$, then there exists $h \in G$ such that $x_2 = h \cdot x_1$, thus the map $(g, y, z) \mapsto (hg, y, z)$ is a Borel embedding of the restriction of E_{x_1} to the set $\{(g, y, z) \in R : x_1 = g \cdot y\}$ into E_{x_2}, and it easily follows that there is a Borel embedding of E_{x_1} into E_{x_2}.

Suppose now, towards a contradiction, that $x_1 \not\preceq x_2$ but there is a measure almost treeable-to-one Borel almost homomorphism $\phi : R \to R$ from E_{x_1} to E_{x_2}. Then the function $\psi : X_{S_{x_1}} \to R$ given by $\psi(z) = (1_G, x_1, z)$ is a

Borel embedding of $E_{S_{x_1}}$ into E_{x_1}. Set $\pi = \phi \circ \psi$. By throwing out a $\mu_{S_{x_1}}$-null, $E_{S_{x_1}}$-invariant Borel subset of $X_{S_{x_1}}$, we can assume that π is of the form $\pi(x) = (g, y, \pi'(z))$, where $\pi' : X_{S_{x_1}} \to X_{S_y}$ is a Borel function and $x_2 = g \cdot y$, thus $x_1 \neq y$. Then π' is a $\mu_{S_{x_1}}$-almost treeable-to-one Borel almost homomorphism from $E_{S_{x_1}}$ to E_{S_y}, which contradicts Corollary 3.3. $\square$

4 The existence of Borel homomorphisms

In this section, we establish a technical result concerning special types of embeddings of E_0. Central to our work here is an elementary observation concerning the definability of non-empty finite sets from certain families of finite sets.

Let $[X]^n$ denote the family of subsets of X of cardinality n. We say that $\mathcal{F} \subseteq [X]^n$ is an *intersecting family* if it is non-empty and

$$\forall S, T \in \mathcal{F} \ (S \cap T \neq \emptyset).$$

For each positive integer $m < n$, define $\mathcal{F}^{(m)} \subseteq [X]^m$ by

$$\mathcal{F}^{(m)} = \{T \in [X]^m : |\{S \in \mathcal{F} : T \subseteq S\}| \geq \aleph_0\}.$$

Proposition 4.1 *Suppose that $\mathcal{F} \subseteq [X]^n$ is an infinite intersecting family. Then there is a positive integer $m < n$ such that $\mathcal{F}^{(m)}$ is an intersecting family.*

Proof. Our assumption that $\mathcal{F}$ is an infinite intersecting family ensures that $\mathcal{F}^{(1)} \neq \emptyset$ and $n \geq 2$. Fix $m < n$ largest such that $\mathcal{F}^{(m)} \neq \emptyset$.

Lemma 4.2 *Suppose that $T \in \mathcal{F}^{(m)}$ and $U \subseteq X$ is finite. Then there exists $S \in \mathcal{F}$ such that $T \subseteq S$ and $S \cap U = T \cap U$.*

Proof. For each $x \in U \setminus T$, the definition of m ensures that there are only finitely many $S \in \mathcal{F}$ for which $T \cup \{x\} \subseteq S$. As $T \in \mathcal{F}^{(m)}$, it therefore follows that there exist infinitely many $S \in \mathcal{F}$ such that $T \subseteq S$ and $S \cap U = T \cap U$. $\square$

To see that $\mathcal{F}^{(m)}$ is an intersecting family, suppose that $T, U \in \mathcal{F}^{(m)}$. Lemma 4.2 ensures that there is a set $S_T \in \mathcal{F}$ such that $T \subseteq S_T$ and $S_T \cap U = T \cap U$, and another application of Lemma 4.2 then ensures that there is a set $S_U \in \mathcal{F}$ such that $U \subseteq S_U$ and $S_U \cap S_T = U \cap S_T = T \cap U$, thus $T \cap U \neq \emptyset$. $\square$

Define $\mathcal{F}^{(s)}$ recursively, for $s \in \mathbb{N}^{<\mathbb{N}}$, by setting $\mathcal{F}^{(\emptyset)} = \mathcal{F}$ and

$$\mathcal{F}^{(sn)} = (\mathcal{F}^{(s)})^{(n)}.$$

Proposition 4.3 *Suppose that $\mathcal{F} \subseteq [X]^n$ is an intersecting family. Then there is a sequence $s \in \mathbb{N}^{<\mathbb{N}}$ such that $\mathcal{F}^{(s)}$ is a finite intersecting family.*

Proof. By induction on n. If $n = 1$, then $|\mathcal{F}| = 1$, thus $s = \emptyset$ is as desired. Suppose now that we have established the proposition up to n, and $\mathcal{F} \subseteq [X]^{n+1}$. If $\mathcal{F}$ is finite, then $s = \emptyset$ is again as desired. Otherwise, Proposition 4.1 ensures that there is a positive integer $m < n + 1$ such that $\mathcal{F}^{(m)}$ is an intersecting family, and the induction hypothesis then ensures that there is a sequence $s \in \mathbb{N}^{<\mathbb{N}}$ such that $(\mathcal{F}^{(m)})^{(s)}$ is a finite intersecting family, thus ms is as desired. □

Recall that the *full group* of E is the group $[E]$ of all Borel automorphisms $\gamma : X \to X$ such that $\text{graph}(\gamma) \subseteq E$. Suppose now that $\phi : X \to Y$ is a Borel function and $\Gamma \subseteq [E]$ is finite. We say that a set $B \subseteq X$ is *(ϕ, Γ)-intersecting* if for each $x \in B$, the set $S_x = \{[\phi(\gamma \cdot y)]_F : \gamma \in \Gamma\}$ is of the same cardinality as Γ and the set $\mathcal{F}_x = \{S_y : y \in [x]_{E|B}\}$ is an intersecting family. We say that B is *ϕ-intersecting* if there is a finite set $\Gamma \subseteq [E]$ such that B is (ϕ, Γ)-intersecting, and we use $\mathcal{I}_\phi$ to denote the σ-ideal generated by ϕ-intersecting Borel sets.

Proposition 4.4 *Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, $\phi : X \to Y$ is Borel, and $X \in \mathcal{I}_\phi$. Then there is a Borel function $\psi : X \to X$ whose graph is contained in E such that $\phi \circ \psi$ is an almost homomorphism from E to F.*

Proof. By breaking X into countably many E-invariant Borel sets, we can assume that there is a finite set $\Gamma \subseteq [E]$ and a (ϕ, Γ)-intersecting Borel set $B \subseteq X$ which intersects every E-class. Proposition 4.3 ensures that for each $x \in B$, there is a sequence $s \in \mathbb{N}^{<\mathbb{N}}$ such that $\mathcal{F}_x^{(s)}$ is a finite intersecting family. The Lusin-Novikov uniformization theorem ensures that, by again breaking X into countably many E-invariant Borel sets, we can assume that there is a single $s \in \mathbb{N}^{<\mathbb{N}}$ such that for every $x \in B$, the set $\mathcal{F}_x^{(s)}$ is a finite intersecting family. Let $A = \{x \in B : [\phi(x)]_F \in \bigcup \mathcal{F}_x^{(s)}\}$, and appeal once more to the Lusin-Novikov uniformization theorem to find a Borel function $\psi : X \to A$ whose graph is contained in E. It is clear that $\phi \circ \psi$ is an almost homomorphism. □

We will now prove a Glimm-Effros style dichotomy theorem:

Theorem 4.5 *Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, and $\phi : X \to Y$ is Borel. Then exactly one of the following holds:*

(1) $X \in \mathcal{I}_\phi$;
(2) There is a continuous embedding $\pi : 2^\mathbb{N} \to X$ of E_0 into E such that

$$\forall \alpha, \beta \in 2^\mathbb{N} \ (\alpha \neq \beta \Rightarrow \neg \phi \circ \pi(\alpha) F \phi \circ \pi(\beta)).$$

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that $X \in \mathcal{I}_\phi$ and there is a Borel embedding $\pi : 2^{\mathbb{N}} \to X$ of E_0 into E such that

$$\forall \alpha, \beta \in 2^{\mathbb{N}} \ (\alpha \neq \beta \Rightarrow \neg \phi \circ \pi(\alpha) F \phi \circ \pi(\beta)).$$

Then $\pi(2^{\mathbb{N}}) \in \mathcal{I}_\phi$, so by Proposition 4.4, there is a Borel function $\psi : \pi(2^{\mathbb{N}}) \to \pi(2^{\mathbb{N}})$ whose graph is contained in $E|\pi(2^{\mathbb{N}})$ such that $\phi \circ \psi$ is an almost homomorphism from E to F. It follows that $\pi^{-1}(\psi \circ \pi(2^{\mathbb{N}}))$ intersects every equivalence class of E_0 in a non-empty finite set, thus E_0 is smooth, the desired contradiction.

It remains to show $\neg(1) \Rightarrow (2)$. Towards this end, suppose that $X \notin \mathcal{I}_\phi$. Fix countable groups Γ and Δ of Borel automorphisms such that $E = E_\Gamma^X$ and $F = E_\Delta^Y$. The usual change of topology arguments allow us to assume that X and Y are zero-dimensional, Γ and Δ act by homeomorphisms, and ϕ is continuous. Fix exhaustive, increasing sequences $\langle \Gamma_n \rangle \in \mathcal{P}(\Gamma)^{\mathbb{N}}$ and $\langle \Delta_n \rangle \in \mathcal{P}(\Delta)^{\mathbb{N}}$ of finite, symmetric neighborhoods of 1_Γ and 1_Δ.

We will recursively find clopen sets $X \supseteq U_0 \supseteq U_1 \supseteq \cdots$ and group elements $\gamma_n \in \Gamma$. Associated with these are the group elements

$$\gamma_s = \prod_{n<|s|} \gamma_n^{s(n)},$$

and the clopen sets $U_s = \gamma_s(U_{|s|})$, for each $s \in 2^{<\mathbb{N}}$, as well as the Borel sets

$$X_n = \{x \in X : \forall s, t \in 2^n \ (s \neq t \Rightarrow \neg \phi(\gamma_s \cdot x) F \phi(\gamma_t \cdot x))\},$$

for each $n \in \mathbb{N}$. We will ensure that the following conditions are satisfied:

(1) $U_n \cap X_n \notin \mathcal{I}_\phi$;
(2) $\forall s \in 2^{n+1}$ $(\mathrm{diam}(U_s) \leq 1/n)$;
(3) $\forall s, t \in 2^n \, \forall \gamma \in \Gamma_n \ (\gamma(U_{s0}) \cap U_{t1} = \emptyset)$;
(4) $\forall s, t \in 2^{n+1} \, \forall \delta \in \Delta_n \ (s \neq t \Rightarrow \delta(\phi(U_s)) \cap \phi(U_t) = \emptyset)$.

We begin by setting $U_0 = X$. Suppose now that we have found $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n$ and $\gamma_0, \gamma_1, \ldots, \gamma_{n-1} \in \Gamma$ which satisfy conditions (1) – (4). For each $\zeta \in \Gamma$, let $U_{n,\zeta}$ denote the set of all $x \in X$ such that:

(a) $x, \zeta \cdot x \in U_n$;
(b) $\forall s, t \in 2^n \, \forall \gamma \in \Gamma_n \ (\zeta \cdot x \neq \gamma_t^{-1} \gamma \gamma_s \cdot x)$;
(c) $\forall s, t \in 2^{n+1} \, \forall \delta \in \Delta_n \ (s \neq t \Rightarrow \delta \cdot \phi(\gamma_{s|n} \zeta^{s(n)} \cdot x) \neq \phi(\gamma_{t|n} \zeta^{t(n)} \cdot x))$.

Define also $X_{n,\zeta} \subseteq X_n$ by

$$X_{n,\zeta} = \{x \in X : \forall s, t \in 2^{n+1} \ (s \neq t \Rightarrow \neg \phi(\gamma_{s|n} \zeta^{s(n)} \cdot x) F \phi(\gamma_{t|n} \zeta^{t(n)} \cdot x))\},$$

and put $B_n = (U_n \cap X_n) \setminus \bigcup_{\zeta\in\Gamma}(U_{n,\zeta} \cap X_{n,\zeta})$.

Lemma 4.6 *$B_n \in \mathcal{I}_\phi$.*

Proof. Note first that B_n is the union of countably many Borel sets $B_{n,k}$ with the property that

$$\forall x \in B_{n,k} \, \forall s,t \in 2^n \, \forall \gamma \in \Gamma_n \; (x \neq \gamma_t^{-1}\gamma\gamma_s \cdot x \Rightarrow \gamma_t^{-1}\gamma\gamma_s \cdot x \notin B_{n,k}).$$

It is clearly enough to show that $\forall k \; (B_{n,k} \in \mathcal{I}_\phi)$. Towards this end, suppose that $x, x' \in B_{n,k}$ are distinct and E-equivalent, and fix $\zeta \in \Gamma$ with $x' = \zeta \cdot x$. As $x \notin U_{n,\zeta} \cap X_{n,\zeta}$, there exist $s,t \in 2^n$ such that $\phi(\gamma_s \cdot x) F \phi(\gamma_t \cdot x')$, and it follows that $B_{n,k}$ is $(\phi, \{\gamma_s : s \in 2^n\})$-intersecting. □

By Lemma 4.6, there exists $\gamma_n \in \Gamma$ such that $U_{n,\gamma_n} \cap X_{n,\gamma_n} \notin \mathcal{I}_\phi$. As U_{n,γ_n} is open, the continuity of ϕ and the actions of Γ and Δ ensures that U_{n,γ_n} is the union of countably many clopen sets $U'_{n,k}$ such that:

(i) $\forall s \in 2^{n+1} \; (\mathrm{diam}(\gamma_s(U'_{n,k})) < 1/n)$;
(ii) $\forall s,t \in 2^n \, \forall \gamma \in \Gamma_n \; (\gamma\gamma_s(U'_{n,k}) \cap \gamma_t\gamma_n(U'_{n,k}) = \emptyset)$;
(iii) $\forall s,t \in 2^{n+1} \, \forall \delta \in \Delta_n \; (s \neq t \Rightarrow \delta(\phi(\gamma_s(U'_{n,k}))) \cap \phi(\gamma_t(U'_{n,k})) = \emptyset)$.

As $\mathcal{I}_\phi$ is a σ-ideal, there exists k such that $U'_{n,k} \cap X_{n,\gamma_n} \notin \mathcal{I}_\phi$. Put $U_{n+1} = U'_{n,k}$.

This completes the recursive construction. Condition (2) ensures that for each $\alpha \in 2^\mathbb{N}$, the decreasing sequence $\langle U_{\alpha|n}\rangle_{n\in\mathbb{N}}$ has vanishing diameter, thus we can define $\pi : 2^\mathbb{N} \to X$ by setting

$$\pi(\alpha) = \text{the unique element of } \bigcap_{n\in\mathbb{N}} U_{\alpha|n}.$$

Conditions (2) and (3) ensure that π is a continuous injection.

To see that $\alpha E_0 \beta \Rightarrow \pi(\alpha) E \pi(\beta)$, it is enough to check the following:

Lemma 4.7 *If $k \in \mathbb{N}$, $s \in 2^k$, and $\alpha \in 2^\mathbb{N}$, then $\pi(s\alpha) = \gamma_s \cdot \pi(0^k\alpha)$.*

Proof. Simply observe that

$$\begin{aligned}\{\pi(s\alpha)\} &= \bigcap_{n\in\mathbb{N}} U_{s(\alpha|n)} \\ &= \gamma_s\Big(\bigcap_{n\in\mathbb{N}} U_{0^k(\alpha|n)}\Big) \\ &= \gamma_s(\{\pi(0^k\alpha)\}),\end{aligned}$$

thus $\pi(s\alpha) = \gamma_s \cdot \pi(0^k\alpha)$. □

To see that $(\alpha,\beta) \notin E_0 \Rightarrow (\pi(\alpha),\pi(\beta)) \notin E$, it is enough check the following:

Lemma 4.8 *Suppose that $\alpha(n) \neq \beta(n)$. Then $\forall \gamma \in \Gamma_n$ $(\gamma \cdot \pi(\alpha) \neq \pi(\beta))$.*

Proof. Suppose, towards a contradiction, that there exists $\gamma \in \Gamma_n$ with $\gamma \cdot \pi(\alpha) = \pi(\beta)$. By the symmetry of Γ_n, we can assume that $\alpha(n) < \beta(n)$. Set $s = \alpha|n$ and $t = \beta|n$, and put

$$x = \gamma_s^{-1} \cdot \pi(\alpha) \text{ and } y = \gamma_n^{-1}\gamma_t^{-1} \cdot \pi(\beta),$$

noting that these are both elements of U_{n+1}. Then $\gamma\gamma_s \cdot x = \gamma_t\gamma_n \cdot y$, which contradicts condition (3). □

It only remains to check that if $\alpha \neq \beta$, then $\neg[\phi \circ \pi](\alpha) F [\phi \circ \pi](\beta)$. This follows from the fact that if $n \in \mathbb{N}$ is sufficiently large that $\alpha|n \neq \beta|n$, then condition (4) ensures that $\forall \delta \in \Delta_n$ $(\delta \cdot [\phi \circ \pi](\alpha) \neq [\phi \circ \pi](\beta))$. □

We are now ready to prove the primary result of the paper:

Theorem 4.9 *Suppose that X and Y are Polish spaces and E and F are countable Borel equivalence relations on X and Y. Then the following are equivalent:*

(1) There is a smooth-to-one Borel almost homomorphism from E to F;
(2) There is a Borel homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$.

Proof. To see (1) ⇒ (2), simply observe that by Remark 2.4, every smooth-to-one Borel almost homomorphism from E to F is necessarily a Borel homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$. To see (2) ⇒ (1), suppose that $\phi : X \to Y$ is a Borel homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$. Then $X \in \mathcal{I}_\phi$, by Theorem 4.5, thus Proposition 4.4 ensures that there is a Borel function $\psi : X \to X$ whose graph is contained in E such that $\phi \circ \psi$ is an almost homomorphism from E to F. As ϕ is a homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$, so too is $\phi \circ \psi$, and it follows that $\pi = \phi \circ \psi$ is a smooth-to-one Borel almost homomorphism from E to F. □

5 Homogeneous σ-ideals

In this section, we obtain our main results concerning the homogeneity of $\mathcal{I}_E$.

Theorem 5.1 *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

(1) E is hyperfinite;
(2) $\mathcal{I}_E$ is homogeneous.

Proof. We begin with (1) ⇒ (2). It is clear that if E is smooth, then $\mathcal{I}_E$ is homogeneous. By Corollary 4.7.6 of Zapletal [17], the σ-ideal $\mathcal{I}_{E_0}$ is homogeneous. By Theorem 7.1 of Dougherty-Jackson-Kechris [3], every non-smooth hyperfinite equivalence relation E is Borel bi-reducible with E_0, thus it follows from Theorem 4.9 that $\mathcal{I}_E$ is homogeneous.

To see $\neg(1) \Rightarrow \neg(2)$, suppose that E is not hyperfinite. By Theorem 1.1 of Harrington-Kechris-Louveau [7], there is a Borel set $B \subseteq X$ such that $E|B$ is non-smooth and hyperfinite. By Remark 2.4 and Theorem 4.9, there is no Borel homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_{E|B}$, thus $\mathcal{I}_E$ is not homogeneous. □

As a corollary, it follows that if E is not hyperfinite and $\langle B_n \rangle_{n \in \mathbb{N}}$ is a cover of X by Borel sets, then there exists $n \in \mathbb{N}$ such that $\mathcal{I}_{E|B_n}$ is not homogeneous. Along similar lines, we have the following:

Theorem 5.2 *Suppose that X and Y are Polish spaces, E is a non-measure hyperfinite equivalence relation on X, F is a measure hyperfinite equivalence relations on Y, and $\pi : X \to Y$ is a Borel homomorphism from E to F. Then there exists $y \in Y$ such that $\mathcal{I}_{E|\pi^{-1}([y]_F)}$ is not homogeneous.*

Proof. Suppose, towards a contradiction, that each of the σ-ideals $\mathcal{I}_{E|\pi^{-1}([y]_F)}$ is homogeneous. Theorem 5.1 then implies that π is hyperfinite-to-one, and Theorem 2.7 ensures that E is measure hyperfinite, a contradiction. □

We are now ready for our final result:

Theorem 5.3 *Every $\mathbf{\Sigma}^1_1$ quasi-order on a Polish space is Borel on Borel reducible to $\preceq_B$, as is every $\mathbf{\Sigma}^1_2$ subset of a Polish space.*

Proof. This follows from Theorems 3.4 and 4.9. □

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