

# BROOKS'S THEOREM FOR MEASURABLE COLORINGS

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## 1. INTRODUCTION

We begin by recalling a classical theorem of Brooks from finite combinatorics:

**Theorem 1.1** (Brooks's Theorem [6, Theorem 5.2.4]). *Suppose  $G$  is a graph on a finite vertex set  $X$  with vertex degree bounded by  $d$ . Suppose further that  $G$  contains no cliques on  $d + 1$  vertices, and if  $d = 2$  that  $G$  contains no odd cycles. Then  $G$  admits a  $d$ -coloring.*

Throughout, by a *graph* we mean a simple undirected graph, where the *degree* of a vertex is its number of neighbors, and a  *$d$ -coloring* is a function assigning each vertex one of  $d$  colors so that adjacent vertices are mapped to different colors.

This paper examines measurable analogues of Brooks's Theorem. While a straightforward compactness argument extends Brooks's Theorem to infinite graphs, such an argument cannot produce a coloring with desirable measurability properties such as being  $\mu$ -measurable with respect to some probability measure, or being Baire measurable with respect to some Polish topology. Indeed, in this setting a straightforward analogue of the  $d = 2$  case of Brooks's Theorem does not hold for either of these measurability notions. Let  $S : \mathbb{T} \rightarrow \mathbb{T}$  be an irrational rotation of the unit circle  $\mathbb{T}$ , and let  $G_S$  be the graph on  $\mathbb{T}$  rendering adjacent each point  $x \in \mathbb{T}$  and its image  $S(x)$  under  $S$ . Then  $G_S$  is acyclic, each vertex has degree 2, and an easy ergodicity argument shows that  $G_S$  has no Lebesgue measurable 2-coloring: since  $S$  is measure preserving, the color sets would have to have equal measure, but since  $S^2$  is ergodic, the color sets would each have to be null or conull. Similarly,  $G_S$  has no Baire measurable 2-coloring (see Section 8).

Our main result is the following measurable analogue of Brooks's theorem for the case  $d \geq 3$ . Recall that a *standard Borel space* is a set  $X$  equipped with a  $\sigma$ -algebra generated by a *Polish* (separable, completely metrizable) topology. Then a *Borel graph*  $G$  is a graph whose vertices are the elements of some standard Borel space  $X$ , and whose edge relation is Borel as a subset of  $X \times X$ .

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**Theorem 1.2.** *Suppose that  $G$  is a Borel graph on a standard Borel space  $X$  with vertex degree bounded by a finite  $d \geq 3$ . Suppose further that  $G$  contains no cliques on  $d + 1$  vertices.*

- (1) *Let  $\mu$  be any Borel probability measure on  $X$ . Then  $G$  admits a  $\mu$ -measurable  $d$ -coloring.*
- (2) *Let  $\tau$  be any Polish topology compatible with the Borel structure on  $X$ . Then  $G$  admits a Baire measurable  $d$ -coloring.*

In the purely Borel setting, Kechris, Solecki, and Todorcevic [9, Proposition 4.6] have shown that every Borel graph of vertex degree bounded by  $d$  admits a Borel  $(d + 1)$ -coloring. Here, Marks [12, Theorem 1.3] has shown that this result is optimal even for acyclic graphs; for every  $d$ , there is an acyclic Borel graph of degree  $d$  with no Borel  $d$ -coloring. Hence, to obtain a measurable analog of Brooks's theorem as in Theorem 1.2, we must consider measurability constraints weaker than Borel measurability. In this vein, Theorem 1.2 improves prior results of Conley and Kechris who proved an analogous result for approximate colorings where one is allowed to discard a set of arbitrarily small measure [5, Theorems 2.19, 2.20].

The proof of Theorem 1.2 first reduces the general statement to the case  $d = 3$ . We then give two different proofs of the  $d = 3$  case of Theorem 1.2. For the first proof, every use of the measure and the topology comes down to one of a few general combinatorial statements, studied in [12], which were shown to hold after discarding a null set or a meager set (see Lemma 2.5). The second proof proceeds by showing the existence of a.e. one-ended spanning subforests of every acyclic Borel graph of degree  $\geq 3$ .

In the case  $d = 2$  we also prove an analogue of Brooks's theorem. However, here we must enlarge our notion of an odd cycle to include the ergodic-theoretic obstruction discussed above.

**Theorem 1.3.** *Suppose  $G$  is a Borel graph on a standard Borel space  $X$  with vertex degree bounded by  $d = 2$  such that  $G$  contains no odd cycles. Let  $E_{2,G}$  be the equivalence relation on  $X$  where  $x E_{2,G} y$  if  $x$  and  $y$  are connected by a path of even length in  $G$ .*

- (1) *Let  $\mu$  be a  $G$ -quasi-invariant Borel probability measure on  $X$ . Then  $G$  admits a  $\mu$ -measurable 2-coloring if and only if there does not exist a non-null  $G$ -invariant Borel set  $A$  such that every  $E_{2,G}$ -invariant Borel subset of  $A$  differs from a  $G$ -invariant set by a nullset.*
- (2) *Let  $\tau$  be a  $G$ -quasi-invariant Polish topology compatible with the Borel structure on  $X$ . Then  $G$  admits a Baire measurable 2-coloring if and only if there does not exist a non-meager  $G$ -invariant Borel set  $A$  so that every  $E_{2,G}$ -invariant Borel subset of  $A$  differs from a  $G$ -invariant set by a meager set.*

The organization of the paper is as follows. In Section 2 we establish notation and gather some background results from descriptive combinatorics. In Section 3 we reduce the proof of Theorem 1.2 to the case  $d = 3$ . The first

proof of the  $d = 3$  case is given in Section 4, and the second proof is given in Section 6 after proving some results on one-ended subforests in Section 5. In Section 7 we apply Theorem 1.2 to graphs arising from group actions, and we apply the methods of Section 5 along with results from probability to obtain factor of IID  $d$ -colorings of Cayley graphs of degree  $d$ , apart from two exceptional cases. In Section 8, we prove Theorem 1.3. In Appendix A, discuss the relationship between Borel colorings and  $\mu$ -measurable and Baire measurable colorings. We discuss quasi-invariant topologies in Appendix B.

## 2. PRELIMINARIES

A *graph*  $G$  on a vertex set  $X$  is a symmetric, irreflexive relation on  $X$ . Given such a graph, we say that two points  $x, y \in X$  are *neighbors* or are *adjacent* in  $G$  if  $x G y$ . A set  $A \subseteq X$  of vertices of  $G$  is *( $G$ -)independent* if for every  $x, y \in A$  it is not the case that  $x G y$ . A  $G$ -independent set  $A$  is said to be a *maximal independent set* if every vertex of  $G$  is either an element of  $A$ , or a neighbor of an element of  $A$ .

A *path* in a graph  $G$  is a sequence of vertices  $x_0 G x_1 G \dots x_n$  containing no repeated vertices. If  $G$  is a graph on  $X$ , then the graph metric  $d_G : X^2 \rightarrow \mathbb{N} \cup \{\infty\}$  on  $G$  maps  $x, y \in X$  to the length of the shortest path connecting  $x$  and  $y$ , if such a path exists. A *cycle* in a graph  $G$  is a sequence of vertices  $x_0 G x_1 G x_2 \dots x_n = x_0$  such that  $n > 2$ , and  $x_i \neq x_j$  for all  $i < j < n$ . We say that a graph is *acyclic* if it does not contain any cycles.

A *Borel graph* is a graph whose vertices are the elements of a standard Borel space  $X$ , and whose edge relation is Borel as a subset of  $X \times X$ . The restriction  $G \upharpoonright A$  of  $G$  to a set  $A \subseteq X$  is the graph on  $A$  obtained by restricting the relation  $G$  to  $A$ . If  $G$  is a Borel graph, and  $A$  is a Borel set, then since  $A$  inherits the standard Borel structure of  $X$ , we see that  $G \upharpoonright A$  is also a Borel graph. A subset  $A \subseteq X$  of vertices of  $G$  is called an  *$n$ -clique* in  $G$  if  $G \upharpoonright A$  is isomorphic to the complete graph on  $n$  vertices.

The *( $G$ -)degree* of a vertex  $x \in X$  of  $G$  is the cardinality of the set  $\{y \in X : x G y\}$ . We say that  $G$  is  *$d$ -regular* if every vertex in  $G$  has degree  $d$ , and  $G$  has degree  $\leq d$  if every vertex has degree  $\leq d$ . Throughout the paper,  $d$  will always be some finite number indicating the maximum degree of a vertex in our graph. If  $G$  is a Borel graph on  $X$  of degree  $\leq d$ , then if  $A \subseteq X$  is a  $G$ -independent Borel set, then there exists a maximal  $G$ -independent Borel set  $A' \supseteq A$ , by the proof of [9, Proposition 4.2]. From this fact one obtains the following result of Kechris, Solecki, and Todorcevic. Recall that a (proper) *( $Y$ -)coloring* of  $G$  is a function  $c: X \rightarrow Y$  such that if  $x G y$ , then  $c(x) \neq c(y)$ .

**Theorem 2.1** (Kechris, Solecki, and Todorcevic [9, Proposition 4.6]). *If  $G$  is a Borel graph of degree  $\leq d$ , then  $G$  admits a Borel  $(d + 1)$ -coloring.*

A *Borel equivalence relation*  $E$  on a standard Borel space  $X$  is a equivalence relation on  $X$  that is Borel as a subset of  $X \times X$ . A Borel equivalence relation  $E$  is said to be *countable* if all of its  $E$ -classes are countable. We say

that  $E$  is *aperiodic* if all of its equivalence classes are infinite. A set  $A \subseteq X$  is  *$E$ -invariant* if for all  $x, y \in X$  if  $x \in A$  and  $xEy$  then  $y \in A$ . If  $G$  is a Borel graph on  $X$  of degree  $\leq d$ , then we denote by  $E_G$  its connectedness relation, which is a countable Borel equivalence relation. We say that a set  $A \subseteq X$  is  *$G$ -invariant* if it is  $E_G$ -invariant, i.e., it is a union of connected components of  $G$ . If  $A \subseteq X$  is a subset of  $X$ , we let  $[A]_G$  note the smallest  $G$ -invariant set containing  $A$ . The Feldman-Moore theorem gives a useful way of decomposing a countable Borel equivalence relation into a union of countably many functions:

**Theorem 2.2** (Feldman-Moore [8, Theorem 1.3]). *Suppose that  $E$  is a countable Borel equivalence relation on a standard Borel space  $X$ . Then there is a set  $\{T_i\}_{i \in \mathbb{N}}$  of Borel automorphisms of  $X$  such that  $E = \bigcup_i \text{graph}(T_i)$ .*

Let  $X$  and  $Y$  be standard Borel spaces. Let  $\mu$  be a Borel probability measure on  $X$ . We say that a function  $f : X \rightarrow Y$  is  *$\mu$ -measurable* if it is measurable for the completion of  $\mu$ . Let  $\tau$  be a compatible Polish topology on  $X$  (by *compatible* we mean that the sigma algebra generated by the  $\tau$ -open sets coincides with the given Borel sigma algebra on  $X$ ). We say that a function  $f : X \rightarrow Y$  is *Baire measurable (with respect to  $\tau$ )* if it is measurable for the sigma algebra of sets which have the Baire property with respect to the completion of  $\tau$ ; the smallest sigma algebra containing the Borel sets and all  $\tau$ -meager sets.

There is an equivalence between the admitting a  $\mu$ -measurable or Baire measurable coloring, and admitting a Borel coloring modulo an invariant null or meager set:

**Proposition 2.3.** *Suppose  $G$  is a Borel graph on a standard Borel space  $X$  whose connected components are countable. Suppose that  $G$  admits some  $n$ -coloring.*

- (1) *Let  $\mu$  be any Borel probability measure on  $X$ . Then  $G$  admits a  $\mu$ -measurable  $n$ -coloring if and only if there is a  $\mu$ -conull  $G$ -invariant Borel set  $A \subseteq X$  such that  $G \upharpoonright A$  has a Borel  $n$ -coloring.*
- (2) *Let  $\tau$  be any Polish topology compatible with the Borel structure on  $X$ . Then  $G$  admits a Baire measurable  $n$ -coloring if and only if there is a comeager  $G$ -invariant Borel set  $A \subseteq X$  such that  $G \upharpoonright A$  has a Borel  $n$ -coloring.*

*Proof.* We begin with the direction  $\Rightarrow$  of 1. Suppose  $c$  is a  $\mu$ -measurable  $n$ -coloring of  $G$ . By the Feldman-Moore theorem, let  $\{T_i\}_{i \in \mathbb{N}}$  be a set of Borel automorphisms of  $X$  so that the connectedness relation  $E_G$  of  $G$  has  $E_G = \bigcup_{i \in \mathbb{N}} T_i$ . Now for each  $i$ , since  $c \circ T_i$  is  $\mu$ -measurable, there is a  $\mu$ -conull set  $A_i$  such that  $c \circ T_i \upharpoonright A_i$  is Borel. Thus, if  $A = [\bigcap_{i \in \mathbb{N}} A_i]_G$ , then  $A$  is  $\mu$ -conull, and  $c \upharpoonright A$  is Borel. This is because for all  $x \in A$ , if  $i$  is least such that  $T_i^{-1}(x) \in A_i$ , then  $c(x) = (c \circ T_i)(T_i^{-1}(x))$ .

The direction  $\Leftarrow$  of 1 is straightforward; given a Borel  $n$ -coloring  $c$  of  $G \upharpoonright A$ , and an arbitrary  $n$ -coloring  $c'$  of  $G \upharpoonright (X \setminus A)$ , then  $c \cup c'$  is a  $\mu$ -measurable coloring of  $G$ .

The proof of part 2 is identical to the above. Simply replace the phrase  $\mu$ -conull with comeager (with respect to  $\tau$ ), and  $\mu$ -measurable with Baire measurable (with respect to  $\tau$ ).  $\square$

While we have stated our main results in terms of the existence of  $\mu$ -measurable and Baire measurable colorings, throughout the paper we will mostly work with the equivalent formulations given by Proposition 2.3 above. Note here that the classical Brooks's theorem shows the existence of the requisite  $d$ -coloring that we will need to apply the above proposition. Note that here we are assuming the axiom of choice.

Suppose  $G$  is a Borel graph on  $X$ , and  $\mu$  is a Borel probability measure on  $X$ . Then we say that  $\mu$  is  $G$ -quasi-invariant if every  $\mu$ -null set is contained in a  $G$ -invariant  $\mu$ -null set. Now if  $G$  has countable connected components, then for every Borel probability measure  $\mu$  on  $X$ , there exists a  $G$ -quasi-invariant Borel probability measure  $\mu'$  on  $X$  such every  $\mu'$ -null set is  $\mu$ -null (that is,  $\mu'$  dominates  $\mu$ ). This follows from the Feldman-Moore theorem by letting  $\{T_i\}_{i \in \mathbb{N}}$  be a set of Borel automorphisms of  $X$  such that  $E_G = \bigcup_i \text{graph}(T_i)$ , and then setting  $\mu'(A) = \sum_{i \geq 1} 2^{-i} \nu(T_i(A))$  (see [8, Section 8]). A key property of a quasi-invariant measure is that if  $A$  is  $\mu'$ -conull, then it contains a  $G$ -invariant  $\mu'$ -conull set. This is because the set  $\{x : x \notin A \wedge x \in [A]_G\}$  is null since it is contained in the complement of  $A$ , and hence its saturation is null.

Similarly, suppose  $G$  is a Borel graph on  $X$ , and  $\tau$  is a compatible Polish topology for  $X$ . Then we say that  $\tau$  is  $G$ -quasi-invariant if every  $\tau$ -meager set is contained in a  $G$ -invariant  $\tau$ -meager set. We show in Appendix B that if  $G$  has countable connected components, then for every compatible Polish topology  $\tau$  on  $X$ , there is a  $G$ -quasi-invariant compatible Polish topology  $\tau'$  such that every  $\tau'$ -meager set is  $\tau$ -meager.

The combination of the above discussion and Proposition 2.3 justifies our assumption from now on that our measures and topologies are quasi-invariant with respect to the graphs we consider. This is because Proposition 2.3 allows us to reformulate Theorems 1.2 and 1.3 to state the existence of a Borel  $d$ -coloring of  $G \upharpoonright A$  for some  $G$ -invariant Borel  $A$  which is conull or comeager. Thus, the assumption of quasi-invariance is harmless since we may always pass to a quasi-invariant measure or topology without adding any new conull or comeager sets. Our assumption of quasi-invariance is helpful because it frees us from talking constantly about invariant sets; assuming quasi-invariance, a null set or meager set of vertices is always contained in a null set or meager set of connected components respectively.

To finish this section, we will recall a lemma from [12] on disjoint complete sections. We then provide some straightforward strengthenings which we will use often in our constructions.

**Lemma 2.4** ([12, Lemma 4.4.1]). *Suppose  $E$  and  $F$  are countable Borel equivalence relations on a standard Borel space  $X$ , so that every  $E$ -class has cardinality  $\geq 3$  and every  $F$ -class has cardinality  $\geq 2$ .*

- (1) *Let  $\mu$  be any Borel probability measure on  $X$ . Then there is set  $A \subseteq X$  such that  $A$  meets  $\mu$ -a.e.  $E$ -class, and its complement meets  $\mu$ -a.e.  $F$ -class.*
- (2) *Let  $\tau$  be any Polish topology compatible with the Borel structure on  $X$ . Then there is a set  $A \subseteq X$  such that  $A$  meets  $\tau$ -comeagerly many  $E$ -classes, and its complement meets  $\tau$ -comeagerly many  $F$ -classes.*

Our next lemma follows from this result.

**Lemma 2.5.** *Let  $E$  and  $F$  be countable Borel equivalence relations on a standard Borel space  $X$ , and let  $\mu$  be a Borel probability measure on  $X$ .*

- (1) *If every  $E$ -class has at least 3 elements and all the  $F$ -classes have at most 2 elements, then there is a Borel set  $A$  that meets  $\mu$ -a.e.  $E$ -class in at least one place, and  $\mu$ -a.e.  $F$ -class in at most one place.*
- (2) *If every  $E$ -class has infinitely many elements, and there is a  $k \in \mathbb{N}$  so that every  $F$ -class has at most  $k$  elements, then there is a Borel set  $A$  that meets  $\mu$ -a.e.  $E$ -class in infinitely many places, and  $\mu$ -a.e.  $F$ -class in at most one place.*
- (3) *For arbitrary  $E$  and  $F$ , there is a Borel set  $A$  that meets  $\mu$ -a.e. infinite  $E$ -class and its complement meets  $\mu$ -a.e. infinite  $F$ -class.*

Moreover, if  $\tau$  is any Polish topology compatible with the Borel structure on  $X$ , then the same statements hold with “ $\mu$ -a.e.” replaced by  $\tau$ -comeagerly many.

*Proof.* For part 1, let  $X'$  be  $\{0, 1\} \times X$ , so that we have the canonical embedding  $\pi: X \rightarrow X'$  where  $\pi(x) = (0, x)$ . Let  $\mu' = \pi_*(\mu)$  be the pushforward of  $\mu$  under  $\pi$  where  $\mu'(A) = \mu(\pi^{-1}(A))$ . Let  $E'$  and  $F'$  be the equivalence relations on  $X'$ , where  $(i, x)E'(j, y)$  if  $i = j$  and  $xEy$ , and  $(i, x)F'(j, y)$  if  $i = j$  and  $xFy$ , or  $i \neq j$ ,  $x = y$  and  $[x]_F$  has one element. Hence, every  $F'$ -class has exactly two elements. Now apply Lemma 2.4 to the equivalence relations  $E'$  and  $F'$ , and  $\mu'$ . We obtain a Borel set  $A' \subseteq X'$  so that  $A'$  meets  $\mu'$ -a.e.  $E'$ -class and its complement meets  $\mu'$ -a.e.  $F'$  class. But then if we let  $A = \pi^{-1}(A')$ , then  $A$  meets  $\mu$ -a.e.  $E$ -class in at least one place, and  $A$  meets  $\mu$ -a.e.  $F$ -class in at most one place, since the complement of  $A'$  meets every  $F'$ -class.

For part 2, first observe that if every  $E$  class has infinitely many elements, and every  $F$  class has at most 2 elements, then there is a Borel set  $A$  that meets  $\mu$ -a.e.  $E$ -class infinitely many times, and  $\mu$ -a.e.  $F$ -class at most once. This is because by [8, Proposition 7.4] there is some  $E' \subseteq E$  such that every  $E'$ -class has 3 elements, so we can apply part (1) to  $E'$  and  $F$ . For the general case, use Theorem 2.2 to obtain finitely many equivalence relations  $F_0, F_1, \dots, F_n$  such every  $F_i$ -class has at most 2 elements, and so that every pair of  $F$ -related points is  $F_i$  related for some  $i \leq n$ . By the

above observation there is some Borel set  $A_0$  which meets  $\mu$ -a.e.  $E$ -class infinitely many times, and which meets  $\mu$ -a.e.  $F_0$  class at most once. Now inductively, for  $i > 0$  there is some Borel set  $A_i$  which meets  $\mu$ -a.e.  $E \upharpoonright A_{i-1}$  class infinitely many times, and  $\mu$ -a.e.  $F_i \upharpoonright A_i$ -class at most once. Take  $A = A_n$ .

For part  $\beta$ , let  $X' = \mathbb{N} \times X$ , let  $\pi: X \rightarrow X'$  be defined by  $\pi(x) = (0, x)$ , and let  $\mu' = \pi_*(\mu)$ . Now let  $E'$  be the equivalence relations on  $X'$  where  $(i, x)E'(j, y)$  if  $xE'y$  and  $i = j$ , or  $xE'y$  and  $[x]_{E'}$  is finite. Similarly, let  $F'$  be the equivalence relations on  $X'$  where  $(i, x)F'(j, y)$  if  $xF'y$  and  $i = j$ , or  $xF'y$  and  $[x]_{F'}$  is finite. Now apply Lemma 2.4 to  $E'$  and  $F'$  to obtain a set  $A' \subseteq X'$  meeting  $\mu'$ -a.e.  $E'$ -class, and so that its complement meets  $\mu'$ -a.e.  $F'$ -class. Then let  $A = \pi^{-1}(A')$ .  $\square$

In Sections 3 through 6, except for Lemma 5.3, the only way in which the measure  $\mu$  and the topology  $\tau$  will be used is via Lemma 2.5. Otherwise, all our constructions in those sections are purely Borel. With this in mind, we will work in just the measure theoretic setting, since the Baire category version of each statement and proof is obtained through a straightforward translation.

### 3. MEETING CLIQUES

In this section, we show that the case  $d = 3$  in Theorem 1.2 implies every other case.

**Lemma 3.1.** *Suppose  $G$  is a Borel graph on a standard Borel space  $X$  of finite bounded degree  $\leq d$ , where  $d \geq 3$ . Suppose further that  $G$  contains no cliques on  $d + 1$  vertices. Let  $\mu$  be any Borel probability measure on  $X$ . Then there is a maximal independent Borel set  $A \subseteq X$  such that  $A$  meets  $\mu$ -a.e.  $d$ -clique contained in  $G$ .*

*Proof.* Suppose  $R, S \subseteq X$  are distinct  $d$ -cliques of  $G$ . Then it is easy to see that either  $R$  and  $S$  are disjoint, or  $R \cap S$  contains exactly  $d - 1$  vertices. If  $R \cap S \neq \emptyset$ , then furthermore we can see that  $R$  and  $S$  are disjoint from every other  $d$ -clique in  $G$ .

Now we construct our independent set meeting  $\mu$ -a.e.  $d$ -clique. First, let  $A$  be a Borel set consisting of one vertex in  $R \cap S$  for every pair of  $d$ -cliques  $R$  and  $S$  with nonempty intersection. By our discussion above, no vertex in  $A$  is adjacent to an element of any  $d$ -clique except the two containing it.

Now let  $Y$  be the set of vertices that are contained in  $d$ -cliques that do not meet any other  $d$ -clique. If  $Y$  is  $\mu$ -null, we are done; our set  $A$  above can be extended in a Borel way to be maximal and independent. Otherwise, equip  $Y$  with the normalized measure  $\nu = (1/\mu(Y))(\mu \upharpoonright Y)$ . We define two Borel equivalence relations  $E$  and  $F$  on  $Y$ . First,  $x E y$  if the unique  $d$ -cliques containing  $x$  and  $y$  are equal. Second,  $x F y$  if  $x = y$  or  $x$  and  $y$  are adjacent in  $G$  and are not  $E$ -related. Clearly, every  $F$  class has cardinality at most 2. Now by Lemma 2.5.1, we see that there exists some  $A'$  such that  $A'$  meets

$\nu$ -a.e.  $E$ -class in exactly one place, and meets  $\nu$ -a.e.  $F$ -class in at most one place. To finish, extend  $A \cup A'$  to a maximal independent Borel set.  $\square$

It is interesting to note that this Lemma is not true in the Borel context for any  $d \geq 1$ . If we equip the group  $\mathbb{Z}/2\mathbb{Z}$  with its usual single generator, and generate the group  $\mathbb{Z}/d\mathbb{Z}$  with all the nonidentity elements of this group, then the Caley graph of their free product  $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/d\mathbb{Z})$  is a  $d$ -regular graph consists of infinitely many  $d$ -cliques connected to each other by single edges. Now in the notation of [12], for the graph  $G(\Gamma, \mathbb{N})$  generated by the free part of the left shift action of this group on  $\text{Free}(\mathbb{N}^\Gamma)$ , there can be no independent Borel set  $A$  meeting every  $d$ -clique by citeMarks\*Theorem 1.6.

Using Lemma 3.1, we can now deduce the Theorem 1.2 from the special case  $d = 3$ .

**Lemma 3.2.** *The case  $d = 3$  of Theorem 1.2 implies every other case.*

*Proof.* We prove part 1. As discussed in Section 2, the proof for Baire category is analogous. Suppose  $G$  is a Borel graph on  $X$  of vertex degree bounded by  $d$ , and suppose  $\mu$  is a Borel probability measure on  $X$ . As discussed in Section 2, we may assume that  $\mu$  is  $G$ -quasi-invariant. Now applying Lemma 3.1 and using the quasi-invariance of  $\mu$ , there must be a  $G$ -invariant conull Borel set  $Y$ , and a maximal independent Borel set  $A \subseteq Y$  such that  $A$  meets every  $d$ -clique of  $G$ . If we color every vertex in  $A$  a single color and remove these vertices from the graph, we obtain a Borel graph  $G \upharpoonright (Y \setminus A)$  of vertex degree  $\leq d - 1$  with no  $d$ -cliques. Now since  $\mu$  is  $G$ -quasi-invariant, we see  $Y \setminus A$  is not  $\mu$ -null, so we can equip  $Y \setminus A$  with the normalization of the measure  $\mu \upharpoonright Y \setminus A$ . Iterating this process until we reach a graph of degree  $\leq 3$  proves the lemma.  $\square$

#### 4. A PROOF OF THE MEASURABLE BROOKS'S THEOREM

In this section, we prove the  $d = 3$  case of Theorem 1.2. We begin with the following lemma:

**Lemma 4.1.** *Suppose  $E$  is an aperiodic countable Borel equivalence relation on a standard Borel space  $X$ , and  $G$  is a Borel graph on  $X$  of finite bounded degree  $\leq d$ . Let  $\mu$  be a Borel probability measure on  $X$ , and  $d_G$  be the graph metric on  $G$ . Then for every  $n$ , there exists a Borel set  $A$  that meets  $\mu$ -a.e.  $E$ -class such that if  $x, y \in A$  and  $x \neq y$ , then  $d(x, y) \geq n$ .*

*Proof.* Let  $H$  be the graph where  $x H y$  if  $x$  and  $y$  are at most distance  $n$  in  $G$ . Let  $\check{H}$  be the dual graph of  $H$  where the vertices of  $\check{H}$  are the edges of  $H$ , and where two such edges are adjacent in  $\check{H}$  if they share a single vertex. Since  $H$  has finite bounded degree, so does  $\check{H}$ . Hence,  $\check{H}$  has a Borel coloring  $c$  with finitely many colors  $\{0, 1, \dots, m\}$  by Theorem 2.1. Let  $F_0, F_1, \dots, F_m$  be the equivalence relations on  $X$  where  $x F_i y$  if  $x = y$  or  $(x, y) \in H$  and  $c(x, y) = i$ . Then the equivalence classes of  $F_i$  all have size

$\leq 2$ , and it is enough for us to prove that there is a Borel set  $A$  that meets  $\mu$ -a.e.  $E$ -class, and so that  $A$  meets each  $F_i$ -class in at most one point.

Now we can apply Lemma 2.5.2 to obtain a Borel  $A_0 \subseteq X$  such that  $A_0$  meets  $\mu$ -a.e.  $E$ -class in infinitely many places, and  $A_0$  meets each  $F_0$  class in at most once place. Then inductively, given  $A_i$ , consider  $E \upharpoonright A_i$  and  $F_{i+1} \upharpoonright A_i$  and use Lemma 2.5.2 to obtain a Borel  $A_{i+1} \subseteq A_i$  such that  $A_{i+1}$  meets  $\mu$ -a.e.  $E \upharpoonright A_i$ -class infinitely many times, and each  $F_{i+1} \upharpoonright A_i$  class at most once. Our desired set is  $A_m$ .  $\square$

Now we prove the special case of Theorem 1.2 when  $d = 3$  and our graph is acyclic.

**Lemma 4.2.** *Suppose that  $G$  is a Borel graph on a standard Borel space  $X$  with degree  $\leq 3$ . Suppose further that  $G$  is acyclic. Then there exists a  $\mu$ -measurable coloring of  $G$  with 3 colors.*

*Proof.* Let  $A$  be a maximal independent Borel set for  $G$ , so  $G \upharpoonright (X \setminus A)$  has degree  $\leq 2$ . Our basic idea is to transform  $A$  into another maximal independent Borel set  $A'$  whose complement we can measurably 2-color. Now to measurably 2-color such a graph  $G \upharpoonright (X \setminus A')$ , it is sufficient for  $A'$  to be such that there is a Borel set  $B$  meeting  $\mu$ -a.e. connected component of component of  $G \upharpoonright (X \setminus A')$  in exactly one place. Given such a set  $B$ , we can then color starting from this set in the usual greedy fashion.

Let  $Y$  be the set of vertices contained in an infinite 2-regular connected component of  $G \upharpoonright (X \setminus A)$ . Let  $E$  be the equivalence relation on  $X$  where  $x E y$  if either  $x$  and  $y$  are elements of  $Y$ , and are in the same connected component of  $G \upharpoonright Y$ , or  $x$  and  $y$  are not in  $Y$ , and  $x$  and  $y$  are in the same connected component of  $G$ . Now by taking a complete section for  $E$  from Lemma 4.1 and intersecting it with  $Y$ , we can find a Borel set  $B$  meeting  $\mu$ -a.e. connected component of  $G \upharpoonright Y$  so that the distance in  $G$  between any two points in  $B$  is greater than 10.

Let  $f: B \rightarrow A$  map each  $x \in B$  to its unique neighbor in  $A$ . Let  $g$  be the function from  $B$  to finite subsets of  $X$  mapping  $x \in B$  to the set of neighbors  $y$  of  $f(x)$  such that  $f(x)$  is the unique neighbor of  $y$  that is in  $A$ . Hence, for each  $x \in B$ , the set  $g(x)$  contains  $x$ , and at most 2 other points. Now we see that for all  $x \in B$ , we have that  $(A \setminus f(x)) \cup (\bigcup g(x))$  is a maximal independent set; it is independent since the elements of  $g(x)$  have no neighbors other than  $f(x)$  in  $A$ , and maximal by the definition of  $g$ . Further, for any  $C \subseteq B$ , we have that  $A_C = (A \setminus f(C)) \cup (\bigcup g(C))$  is also a maximal independent subset of  $G$ , since elements of  $B$  are far apart. Our plan is to use Lemma 2.5 to find a Borel  $C \subseteq B$  so that  $G \upharpoonright (X \setminus A_C)$  has a  $\mu$ -measurable 2-coloring.

Now for any  $C \subseteq B$ , each 2-regular connected component of  $G \upharpoonright (X \setminus A_C)$  must either be a 2-regular connected component of  $G \upharpoonright (X \setminus A)$ , or a 2-regular connected component of  $G \upharpoonright (X \setminus A_B)$ . Further, each 2-regular connected component of  $G \upharpoonright (X \setminus A_C)$  must contain at least one element of  $B$  or of  $f(B)$ . To see this, note  $A \setminus A_C = f(C)$  contains only points in  $\text{ran}(f)$ .

Hence, any 2-regular connected components in  $G \upharpoonright (X \setminus A_C)$  not contained in  $(X \setminus A)$  must consist of connected components from  $G \upharpoonright (X \setminus A)$  that were “joined up” when we removed some points of the form  $f(x)$  from  $A$ .

Now let  $E$  be the equivalence relation on  $B$  where  $x E y$  if  $x$  and  $y$  are in the same connected component of  $G \upharpoonright (X \setminus A)$ . Note  $E$  is aperiodic. Let  $F$  be the equivalence relation on  $B$  where  $x F y$  if  $f(x)$  and  $f(y)$  are in the same connected component of  $G \upharpoonright (X \setminus A_B)$ . Now by Lemma 2.5.3 let  $C \subseteq B$  be a Borel set such that  $C$  meets  $\mu$ -a.e.  $E$ -class, and its complement meets  $\mu$ -a.e. infinite  $F$ -class. Then each 2-regular connected components of  $G \upharpoonright (X \setminus A_C)$  must meet only finitely many elements of  $B$ , or finitely many elements of  $f(B)$ .

Now if we let  $B' = B \cup \{x \in X \setminus A_C : x \text{ has degree } \leq 1 \text{ in } G \upharpoonright (X \setminus A_C)\}$ , then  $B'$  is a Borel set meeting  $\mu$ -a.e. connected component of  $G \upharpoonright (X \setminus A_C)$ , and meeting each connected component in only finitely many places. Recall from Section 2 that we may assume that there is a Borel linear ordering of  $X$ . From this we can obtain a set  $B'' \subseteq B'$  meeting  $\mu$ -a.e. connected component exactly once by letting  $B''$  be the set of elements of  $B'$  that are leftmost in their connected component of  $G \upharpoonright (X \setminus A_C)$ . Finally, we can obtain a  $\mu$ -measurable 2-coloring of  $G \upharpoonright (X \setminus A_C)$  by using one color for vertices an even distance from  $B''$ , and the other color for vertices an odd distance from  $B''$ .  $\square$

Before we move to the full theorem, we need two more easy lemmas:

**Lemma 4.3.** *Suppose  $G$  is a Borel graph of finite bounded degree  $\leq d$ , and in every connected component of  $G$  there is a vertex of degree  $< d$ . Then  $G$  admits a Borel coloring  $d$ -coloring.*

*Proof.* This lemma is special case of a more general fact we will prove in Section 5.

Let  $X$  be the set of vertices of  $G$ . Let  $f : X \rightarrow \mathbb{N}$  be the function assigning to each vertex  $x$  the distance  $f(x)$  from  $x$  to a vertex of degree  $< d$ . Note that if  $f(x) > 0$ , then  $x$  has at least one neighbor  $y$  such that  $f(y) < f(x)$ .

Let  $A_0 = \emptyset$  and  $c_0$  be the empty function. Iteratively, given  $A_i$  and a Borel  $d$ -coloring  $c_i$  of  $G \upharpoonright A_i$ , let

$$A_{i+1} = A_i \cup \{x : f(x) \geq f(y) \text{ for all } y \in N(x) \text{ with } y \notin A_i\}.$$

Now each  $x \in A_{i+1} \setminus A_i$  has at most  $d - 1$  neighbors in  $A_{i+1}$ ; either  $f(x) = 0$  so  $x$  has degree  $< d$ , or  $f(x) > 0$  in which case there is some neighbor  $y$  of  $x$  with  $f(y) < f(x)$ , and inductively  $y$  cannot be in  $A_{i+1}$ . Hence, we can partition  $A_{i+1} \setminus A_i$  into  $d$  many maximal  $G$ -independent Borel set  $B_0, \dots, B_{d-1}$ . We can then extend  $c_i$  to  $c_{i+1}$ , a Borel  $d$ -coloring of  $G \upharpoonright A_{i+1}$ , by coloring the elements of  $B_0, \dots, B_{d-1}$  in order, coloring  $x \in B_i$  the least color not already used by one of its neighbors. Now let  $A_\infty = \bigcup A_i$  and  $c_\infty = \bigcup c_i$ .

Notice that each  $x \in X \setminus A_\infty$  has at least one neighbor  $y$  in  $X \setminus A_\infty$  such that  $f(y) > f(x)$ . Hence, if we let  $C_0 = A_\infty$ , and  $C_{i+1} = C_i \cup \{x : f(x) \leq i\}$ ,

then every  $x \in C_{i+1} \setminus C_i$  has at most  $d-1$  neighbors in  $B_{i+1}$ . Then as above, we can recursively extend our Borel  $d$ -coloring of  $C_0$  to a Borel  $d$ -coloring of each  $C_i$ . We are done then, since  $X = \bigcup_i C_i$ .  $\square$

We need one more trivial lemma from classical graph theory. In what follows, we will say that a *neighbor of a cycle* is a neighbor of one of the vertices of the cycle which is not contained in the cycle.

**Lemma 4.4.** *Suppose that  $G$  is a graph containing a cycle  $c$ , where each vertex of the cycle has a unique neighbor not in  $c$ . Then given a 3-coloring of the neighbors of  $c$  which does not take a constant value, we can extend this to a 3-coloring that includes the cycle  $c$ .*

*Proof.* Take two adjacent elements  $x$  and  $y$  in the cycle whose neighbors not in  $c$  are assigned different colors. Then color  $x$  the same color as the neighbor of  $y$ , and then color the vertices starting from  $x$  one by one, finishing with  $y$ . At the end of this process we can color  $y$  since two of its neighbors are assigned the same color.  $\square$

We're now ready to finish.

*Proof of Theorem 1.2.* We begin by noting that we can make a number of assumptions about our graph. First, we may assume our graph has degree  $\leq 3$  by Lemma 3.2. Additionally, by Lemma 4.2, we may assume that the graph contains a cycle in each connected component. Indeed, by splitting the graph into countably many invariant pieces, we may assume there is a single number  $k \in \mathbb{N}$  so that there is an cycle of length  $k$  in each connected component, and there are no cycles of length  $< k$ . Then, given any cycle  $c$  of length  $k$ , and vertex  $x$  in  $c$ , there is a unique neighbor of  $x$  not in  $c$ , otherwise  $c$  is not of minimal length.

Now let  $C$  be a Borel set of cycles of length  $k$  that contains at least one cycle from each connected component of  $G$ , and where distinct cycles are disjoint and of distance  $\geq 3$  apart (i.e. vertices in distinct cycles are of distance  $\geq 3$ ). To obtain such a set  $C$ , consider the standard Borel space  $Z$  of all cycles of length  $k$ , and form a graph  $H$  on  $Z$  where two cycles are adjacent if two of their vertices are of distance  $\leq 2$ . Now since  $H$  has finite bounded degree, there is a Borel coloring of  $H$  with finitely many colors. Let  $C$  be the set of cycles that receive the least color assigned to any cycle from the same connected component. Let  $Y$  be the set of points contained in the cycles in  $C$ .

We define two equivalence relations on  $Y$ . First,  $x E y$  if  $x'$  and  $y'$  are in the same connected component of  $G \upharpoonright (X \setminus Y)$  where  $x'$  is the unique neighbor of  $x$  in not in  $Y$ , and  $y'$  is the unique neighbor of  $y$  in not in  $Y$ . Second,  $x F y$  if  $x$  and  $y$  are contained in the same cycle in  $C$ . Now by Lemma 2.5.2, let  $A$  be a Borel set that meets  $\mu$ -a.e. infinite  $E$ -class at least once, and that meets every  $F$ -class at most once. Let  $G'$  be the graph on  $X \setminus Y$  obtained by adding edges to  $G \upharpoonright (X \setminus Y)$  as follows. For each cycle

$c \in C$ , pick two elements  $x, y \in c$  not in  $A$ , and let  $x'$  and  $y'$  be the unique neighbors of  $x$  and  $y$  not contained in  $c$ . If  $x' \neq y'$ , connect  $x'$  to  $y'$  in  $G'$ . If  $x' = y'$ , then pick any other neighbor  $z'$  of  $c$  not equal to  $x'$ , and connect  $z'$  to  $x'$  in  $G'$ . In this latter case, note that  $x'$  has degree 2 in  $G'$ . Note now that two neighbors of every cycle  $c \in C$  in  $G$  are adjacent in  $G'$ , and  $G'$  has degree  $\leq 3$ .

We claim there is a  $\mu$ -measurable 3-coloring of  $G'$ . To see this, let  $Z$  be the set of vertices contained in connected components of  $G'$  that are adjacent to infinitely many cycles in  $C$  (as viewed in our original graph  $G$ ). Now  $\mu$ -a.e connected component of  $G' \upharpoonright Z$  contains at least one vertex of degree  $< 3$ . This is because each such connected component corresponds to an infinite equivalence class of  $E$ , and hence there is some vertex  $x$  in this connected component adjacent to an element of  $A$  in  $G$ . This vertex either has degree 2 in  $G'$  because no edges incident to it were added to  $G'$ , or has a neighbor of degree 2, since then this vertex is  $z'$  in the above discussion. Thus, there is a  $\mu$ -measurable 3-coloring of  $G' \upharpoonright Z$ . Now the remaining connected components of  $G'$  not in  $Z$  are adjacent to only finitely many cycles in  $C$ . Hence, we can find a Borel set meeting each connected component of  $G' \upharpoonright ((X \setminus Y) \setminus Z)$  exactly once, and hence a Borel 3-coloring of this graph. Thus, there is a  $\mu$ -measurable 3-coloring of all of  $G'$ .

Now we claim we can extend this  $\mu$ -measurable 3-coloring of  $G'$  to a  $\mu$ -measurable 3-coloring of  $G$ . This is by Lemma 4.4; for every remaining cycle of  $C$ , two neighbors of the cycle are assigned different colors by our coloring of  $G'$ , since they are connected in  $G'$ .  $\square$

## 5. ONE-ENDED SUBFORESTS

This section focuses on definably isolating certain acyclic subgraphs of various classes of graphs, which will subsequently provide a skeleton along which to construct a coloring. Given a function  $f: X \rightarrow X$ , we define the *forward orbit* of  $x \in X$ , denoted  $f^{+\mathbb{N}}(x)$ , to be the set  $\{f^n(x) : n \in \mathbb{N}\}$ . Analogously, we define the *backward orbit* of  $x$ , denoted  $f^{-\mathbb{N}}(x)$ , to be  $\bigcup_{n \in \mathbb{N}} f^{-n}(x)$ . Unsurprisingly,  $y \in f^{+\mathbb{N}}(x)$  if and only if  $x \in f^{-\mathbb{N}}(y)$ .

It will be useful to define analogous notions for partial functions. Given a subset  $B \subseteq X$  and a function  $f: B \rightarrow X$ , we may define the backward orbit  $f^{-\mathbb{N}}(x)$  exactly as before. We say such a function is *aperiodic* if for all  $x \in X$  and  $n > 0$  we have  $f^n(x) \neq x$ . For functions with full domain this is equivalent to  $f^{+\mathbb{N}}(x)$  being infinite for all  $x$ . We say a function  $f: B \rightarrow X$  has *one end* if it is aperiodic and  $f^{-\mathbb{N}}(x)$  is finite for all  $x$ . The orbits of a one-ended function come in two types. Any such orbit is either finite and contains a unique element of  $X \setminus B$  (and indeed  $f$  “points” every vertex towards this element) or is infinite and each vertex is incident with a unique injective infinite  $G$ -ray (and again  $f$  points towards this ray).

**Proposition 5.1.** *Suppose that  $G$  is a locally finite Borel graph on a standard Borel space  $X$ , and  $A \subseteq X$  is Borel. Then there is a one-ended Borel function  $f: [A]_{E_G} \setminus A \rightarrow [A]_{E_G}$  whose graph is contained in  $G$ .*

*Proof.* Without loss of generality we may assume that  $[A]_{E_G} = X$ . Let  $B$  be the set of  $x \in X \setminus A$  such that there exists an injective  $G$ -ray  $(x_i) \in X^{\mathbb{N}}$  with  $x_0 = x$  and  $d_G(x_{i+1}, A) > d_G(x_i, A)$  for all  $i \in \mathbb{N}$ ; note that König's lemma ensures that  $B$  is Borel. Let  $f: X \setminus A \rightarrow X$  be a Borel function so that

- (1)  $f(x) \in G_x \cap B$  with  $d(f(x), A) > d(x, A)$  if  $x \in B$ ,
- (2)  $f(x) \in G_x$  with  $d(f(x), A) < d(x, A)$  if  $x \notin B$ .

To see that  $f$  is as desired, suppose first that  $x \notin B$ . Then  $f^{-\mathbb{N}}(x) \subseteq X \setminus B$ , and if  $f^{-\mathbb{N}}(x)$  were infinite an application of König's lemma would allow the construction of an injective  $G$ -ray as in the definition of  $B$ , contradicting the fact that  $x \notin B$ . On the other hand, if  $x \in B$  then  $f^{-\mathbb{N}}(x) \cap B$  is finite and is in fact contained in  $\bigcup_{i < d(x, A)} f^{-i}(x)$ . Consequently  $f^{-\mathbb{N}}(x)$  is the union of this finite set with  $\bigcup_{i \leq d(x, A)} \{f^{-i}(y) : y \in f^{-i}(x) \setminus B\}$ , which by the previous case is a finite union of finite sets.  $\square$

An appropriate iteration of Proposition 5.1 allows us to find within certain graphs a one-ended Borel function with conull domain.

**Definition 5.2.** We say that a graph  $G$  is *ample* if every vertex is of  $G$ -degree at least 2, every connected component of  $G$  contains a vertex of  $G$ -degree at least 3, and moreover if  $\deg_G(x) \geq 3$  then each component of  $G \upharpoonright (X \setminus \{x\})$  again contains a vertex of  $G$ -degree at least 3.

Geometrically, an acyclic graph  $G$  is ample if it can be obtained from an acyclic graph with each vertex of degree at least 3 by subdividing each edge into some vertices of degree 2.

**Lemma 5.3.** *Suppose that  $G$  is a bounded-degree, acyclic Borel graph on a standard Borel space  $X$ . Suppose moreover that  $G$  is ample.*

- (1) *Let  $\mu$  be a Borel probability measure on  $X$ . Then there is a  $\mu$ -conull Borel set  $B$  and a one-ended Borel function  $f: B \rightarrow X$  whose graph is contained in  $G$ .*
- (2) *Let  $\tau$  be a compatible Polish topology on  $X$ . Then there is a  $\tau$ -comeager Borel set  $B$  and a one-ended Borel function  $f: B \rightarrow X$  whose graph is contained in  $G$ .*

*Proof.* We prove 1. and then indicate the changes needed for 2. at the end of the proof. Fix  $d \in \mathbb{N}$  bounding the degree of vertices of  $G$  (so  $d \geq 3$ ). The heart of the construction rests in the following claim.

**Claim.** *There is a Borel subset  $A \subseteq X$  meeting each connected component of  $G$  and with  $\mu(A) \leq 1 - d^{-3}$ , such that  $G \upharpoonright A$  is ample.*

*Proof of the claim.* Let  $X_3 = \{x \in X : \deg_G(x) \geq 3\}$  and define an auxiliary graph  $G'$  on  $X_3$  by putting  $x G' y$  if  $x E_G y$  and the unique injective  $G$ -path from  $x$  to  $y$  contains no other points of  $X_3$ . Define a Borel map  $\pi: X \rightarrow X_3$  selecting for each  $x$  a closest element of  $X_3$  with respect to the graph metric in  $G$ . Define then a measure  $\nu$  on  $X_3$  by  $\nu = \pi_*\mu$ , i.e.,  $\nu(B) = \mu(\pi^{-1}(B))$  for all Borel  $B$ .

Finally, let  $H$  be the distance  $\leq 3$  graph associated with  $G'$ , so two distinct points of  $X_3$  are  $H$  related if they are connected by a  $G'$  path of length at most 3. So  $H$  has degree bounded by  $d^3 - 1$ , and hence by Theorem 2.1 a Borel coloring in  $d^3$  colors. Consequently, there is an  $H$ -independent Borel set  $C \subseteq X_3$  with  $\nu(C) \geq d^{-3}$ .

Define  $C' \subseteq X$  by  $x \in C'$  if  $x \in C$  or  $x \in X \setminus X_3$  and can be connected to a point in  $C$  without using any other points of  $X_3$ . Note that  $\pi^{-1}(C) \subseteq C'$ , so in particular  $\mu(C') \geq d^{-3}$ . We then set  $A = X \setminus C'$ , and check that  $A$  satisfies the conclusion of the claim.

The  $G'$ -independence of  $C$  (in conjunction with the ampleness of  $G$ ) implies that  $A$  meets each  $G$  component. The only thing remaining to check is that  $G \upharpoonright A$  is ample. Note that the only way a vertex  $x$  in  $X_3$  can have  $(G \upharpoonright A)$ -degree less than three is if it is  $G'$ -adjacent to an element of  $C$ . So the fact that distinct points of  $C$  have  $G'$  distance at least four implies that  $x$  has two  $G'$  neighbors in  $X_3$  whose  $(G \upharpoonright A)$ -degree remains 3. In particular, the degree of  $x$  is two. Moreover, if  $x$  were used to witness the ampleness condition of one of its neighbors, the condition can be witnessed instead by the other neighbor. So  $G \upharpoonright A$  is ample and the claim is proved.  $\square$

By iterating the claim, we may build a decreasing sequence  $(A_i)_{i \in \mathbb{N}}$  of Borel sets so that  $A_0 = X$ ,  $A_{i+1}$  meets each component of  $G \upharpoonright A_i$ , and  $\mu(\bigcap_i A_i) = 0$ . Apply Proposition 5.1 to find an aperiodic Borel function  $f_i: (A_i \setminus A_{i+1}) \rightarrow A_i$  whose graph is contained in  $G$  such that  $f^{-\mathbb{N}}(x)$  is finite for each  $x \in X$ . Finally, put  $B = X \setminus \bigcap_i A_i$ , and then the function  $f: B \rightarrow X$  defined by  $f = \bigcup_i f_i$  is as desired.

For part 2., the statement to prove in place of the above claim is the following: (\*) For any non-empty  $\tau$ -open set  $U$  there is a Borel subset  $A \subseteq X$  meeting each connected component of  $G$  and with  $U \setminus A$  non-meager, such that  $G \upharpoonright A$  is ample. The proof of (\*) is the same as the proof of the claim except that we choose the  $H$ -independent Borel set  $C \subseteq X_3$  with  $U \cap \pi^{-1}(C)$  non-meager. Then we fix a countable base  $\{U_k\}_{k \in \mathbb{N}}$  of open sets for  $\tau$  and, as in part 1., we iteratively apply (\*) to build a decreasing sequence  $(A_i)_{i \in \mathbb{N}}$  of Borel sets so that  $A_0 = X$ ,  $A_{i+1}$  meets each component of  $G \upharpoonright A_i$ , and with  $U_i \setminus A_i$  non-meager. It follows that  $U_k \setminus \bigcap_i A_i$  is non-meager for all  $k \in \mathbb{N}$ , and therefore  $\bigcap_i A_i$  is meager. The rest of the proof is as before.  $\square$

At last, we use the ability to find one-ended functions inside a graph to help definably color the graph.

**Proposition 5.4.** *Suppose that  $G$  is a Borel graph on a standard Borel space  $X$  with degree bounded by  $d$ . Suppose moreover  $B$  is a Borel subset of  $X$  and that  $f: B \rightarrow X$  is a one-ended Borel function whose graph is contained in  $G$ . Then there is a Borel  $d$ -coloring of  $G \upharpoonright B$ .*

*Proof.* For convenience, extend  $f$  to have full domain  $X$  by fixing points of  $X \setminus B$ . After doing so, we have  $\bigcap_i f^i[X] = X \setminus B$ . Define Borel sets  $X_i = f^i[X] \setminus f^{i+1}[X]$ , so  $B = \bigsqcup_i X_i$ . Moreover, each vertex  $x \in X_i$  is  $G$ -adjacent to at least one vertex in  $X_{i+1}$ , namely,  $f(x)$ .

In particular,  $G \upharpoonright X_0$  has degree bounded by  $d - 1$ , so by Theorem 2.1 there is a Borel coloring  $c_0: X_0 \rightarrow d$  of  $G \upharpoonright X_0$ . The following lemma is a special case of [5, Lemma 2.18], but we include its short proof in the interest of self-containment.

**Lemma 5.5.** *Any Borel coloring  $c_{i-1}: \bigcup_{j < i} X_j \rightarrow d$  of  $G \upharpoonright \bigcup_{j < i} X_j$  extends to a Borel coloring  $c_i: \bigcup_{j \leq i} X_j \rightarrow d$  of  $G \upharpoonright \bigcup_{j \leq i} X_j$ .*

*Proof of the lemma.* Again by Theorem 2.1 there is a partition  $X_i = X_i^1 \sqcup \dots \sqcup X_i^d$  of  $X_i$  into Borel  $G$ -independent sets. First extend  $c_{i-1}$  to a coloring  $c': \bigcup_{j < i} X_j \cup X_i^1$  by coloring each vertex in  $X_i^1$  the least color not used among its (fewer than  $d$  many) colored neighbors. Similarly extend in turn to  $X_i^2, \dots, X_i^d$ .  $\square$

Now iteratively apply the lemma to obtain a coherent sequence of colorings  $c_i: \bigcup_{j \leq i} X_j \rightarrow d$ . Then  $c = \bigcup_{i \in \mathbb{N}} c_i$  is the desired Borel coloring of  $G \upharpoonright B$ .  $\square$

**Corollary 5.6.** *Suppose that  $G$  is a Borel graph on a standard Borel space  $X$  with degree bounded by  $d$ . Suppose moreover that  $f: X \rightarrow X$  is a function with one end whose graph is contained in  $G$ . Then there is a Borel  $d$ -coloring of  $G$ .*

## 6. A SECOND PROOF

In this section we give another proof of Theorem 1.2. Combining Propositions 5.1 and 5.4 allow us to color graphs when certain nice complete sections can be found. Say that a Borel set  $A \subseteq X$  is *3-flexible* if any Borel 3-coloring of  $G \upharpoonright (X \setminus A)$  extends to a Borel 3-coloring of  $A$ . If  $G \upharpoonright A$  has finite connected components, this becomes a purely (finite) combinatorial question: does every 3-coloring of the  $G$ -neighbors of  $A$  extend to a coloring of  $A$ .

**Proposition 6.1.** *Suppose that  $G$  is a graph containing no 4-clique. If each connected component of  $G \upharpoonright A$  is either an isolated vertex of  $G$ -degree less than 3 or a minimal pair of two intersecting cycles, then  $A$  is 3-flexible.*

We now give another proof of Theorem 1.2.

*Proof of Theorem 1.2.* By Lemma 3.2, we may assume  $G$  has degree  $\leq 3$ . Suppose that  $A_0$  is a Borel  $G$ -independent set of vertices with  $G$ -degree less

than 3. Then Propositions 5.1 and 5.4 allow us to find a Borel 3-coloring of  $G \upharpoonright ([A_0]_{E_G} \setminus A_0)$ , and 3-flexibility of  $A_0$  allows us to extend to a Borel 3-coloring of  $G \upharpoonright [A_0]_{E_G}$ . Similarly, if  $A_1$  is a Borel  $G$ -independent set of minimal pairs of intersecting cycles, we may obtain a Borel 3-coloring of  $G \upharpoonright [A_1]_{E_G}$ .

So it suffices to handle the case in which every vertex has degree 3 and distinct cycles are disjoint. Following [8, proof of Corollary 10.2] we first reduce to the case of a quasi-invariant measure. Fix by Theorem 2.2 a sequence  $(T_i)_{i \in \mathbb{N}}$  of Borel automorphisms such that  $E_G = \bigcup_i \text{graph}(T_i)$ . Put  $\nu = \sum_i 2^{-i} (T_i)_* \mu$ . Note that  $\nu$  is  $G$ -quasi-invariant, in the sense that the  $E_G$ -saturation of a  $\nu$ -null set remains  $\nu$ -null. Furthermore, the  $G$ -invariant  $\mu$ -null sets are exactly the  $G$ -invariant  $\nu$ -null sets.

Now let  $G'$  be a Borel subgraph of  $G$  obtained by deleting exactly one edge from every cycle. Then  $G'$  is an acyclic graph which is easily confirmed to be ample. Then, applying Lemma 5.3, we obtain a  $\nu$ -conull Borel set  $B$  and an aperiodic Borel function  $f: B \rightarrow X$  whose graph is contained in  $G'$  (and thus  $G$ ) such that  $f^{-\mathbb{N}}(x)$  is finite for each  $x \in X$ . By quasi-invariance, we may assume that  $B$  is  $G$ -invariant, and hence  $f$  has range contained in  $B$ , and in particular has one end. We then apply Corollary 5.6 to obtain the desired Borel 3-coloring of  $G \upharpoonright B$ .  $\square$

## 7. APPLICATIONS TO GROUP ACTIONS

We consider now (almost everywhere) free, measure-preserving actions of a finitely generated group  $\Gamma$  on a standard probability space  $(X, \mu)$ . Denote by  $\text{FR}(\Gamma, X, \mu)$  the set of such actions. With each  $a \in \text{FR}(\Gamma, X, \mu)$  and finite, symmetric generating set  $S$  of  $\Gamma$  not containing the identity we may associate a graph  $G(S, a)$  on  $X$  by declaring  $x$  and  $y$  adjacent if there exists  $s \in S$  with  $s \cdot x = y$ . Freeness of the action implies that almost every connected component of  $G(S, a)$  is isomorphic to the Cayley graph  $\text{Cay}(\Gamma, S)$ .

In [4, Theorem 6.1] it is shown that for finitely generated infinite groups  $\Gamma$ , any  $a \in \text{FR}(\Gamma, X, \mu)$  is weakly equivalent to some  $b \in \text{FR}(\Gamma, X, \mu)$  whose associated graph  $G(S, b)$  is measure-theoretically  $|S|$ -colorable. Theorem 1.2 eliminates the need to pass to a weakly equivalent action for almost all groups.

**Corollary 7.1.** *Suppose that  $\Gamma$  is an infinite group with finite, symmetric generating set  $S$  such that  $|S| \geq 3$ . Then for any  $a \in \text{FR}(\Gamma, X, \mu)$  the graph  $G(S, a)$  admits a Borel  $|S|$ -coloring on a conull set.*

**Remark 7.2.** The only infinite groups with generating sets  $S$  satisfying  $|S| < 3$  are  $\mathbb{Z}$  with  $S = \{\pm 1\}$  and  $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b \mid a^2 = b^2 = \text{id} \rangle$  with  $S = \{a, b\}$ . Indeed, no graph associated with a free mixing action of either group admits a Borel 2-coloring on a conull set.

Finally, the methods of section 5 may be used in conjunction with some techniques from probability to improve known bounds on the colorings of

Cayley graphs attainable by factors of IID. We consider the *Bernoulli shift* action of a countable group  $\Gamma$  on the space  $[0, 1]^\Gamma$  equipped with product Lebesgue measure  $\mu$ , where  $\gamma \cdot x(\delta) = x(\gamma^{-1}\delta)$ . Denote by  $G(\Gamma, S)$  the graph associated with the Bernoulli shift and generating set  $S$ . For convenience we sometimes work instead with the shift action of  $\Gamma$  on  $[0, 1]^E$ , where  $E$  is the edge set of the (right) Cayley graph  $\text{Cay}(\Gamma, S)$  (and as usual  $\Gamma$  acts by left translation on the Cayley graph). We denote the corresponding graph on  $[0, 1]^E$  by  $G'(\Gamma, S)$ . Since the shift action on  $[0, 1]^E$  is measure-theoretically isomorphic to the Bernoulli shift on  $[0, 1]^\Gamma$ , we lose nothing by working with  $G'(\Gamma, S)$  rather than  $G(\Gamma, S)$ .

We may use each  $x \in [0, 1]^E$  to label the edges of its connected component in  $G'(\Gamma, S)$ , assigning  $(\gamma \cdot x, s\gamma \cdot x)$  the label  $x(\gamma^{-1}, \gamma^{-1}s^{-1})$ . The structure of the action ensures that this labeling is independent of the particular choice of  $x$ , and in particular this labeling is a Borel function from  $G'(\Gamma, S)$  to  $[0, 1]$ . Following [11] we obtain the *wired minimal spanning forest*,  $\text{WMSF}(G'(\Gamma, S))$ , by deleting those edges from  $G'(\Gamma, S)$  which receive a label which is maximal in some simple cycle or bi-infinite path. By construction,  $\text{WMSF}(G'(\Gamma, S))$  is acyclic.

**Theorem 7.3** (Lyons-Peres-Schramm). *Suppose that  $\Gamma$  is a nonamenable group with finite symmetric generating set  $S$ , and consider the graph  $G'(\Gamma, S)$  defined above. There is a conull,  $G'(\Gamma, S)$ -invariant Borel set  $B \subseteq [0, 1]^E$  on which each connected component of  $\text{WMSF}(G'(\Gamma, S))$  has one end.*

*Proof.* See [11, 3.12], which says  $\text{WMSF}(G'(\Gamma, S))$  is almost surely one-ended provided the Cayley graph of  $\Gamma$  has no infinite clusters at critical percolation. This holds for nonamenable Cayley graphs by [2, Theorem 1.1].  $\square$

Let  $\text{Aut}_{\Gamma, S}$  be the automorphism group of the Cayley graph  $\text{Aut}(\text{Cay}(\Gamma, S))$ . Given a group  $\Gamma$  with generating set  $S$  and a natural number  $k$ , we may view the space  $\text{Col}(\Gamma, S, k)$  of  $k$ -colorings of the (right) Cayley graph  $\text{Cay}(\Gamma, S)$  as a closed (thus Polish) subset of  $k^\Gamma$ . The action of  $\Gamma$  by left translations on  $\text{Cay}(\Gamma, S)$  induces an action on  $\text{Col}(\Gamma, S, k)$ . An *automorphism-invariant random  $k$ -coloring* of  $\text{Cay}(\Gamma, S)$  is a Borel probability measure on  $\text{Col}(\Gamma, S, k)$  invariant under this  $\text{Aut}_{\Gamma, S}$  action. Such a random  $k$ -coloring is a factor of IID if it is a factor of the shift action of  $\text{Aut}_{\Gamma, S}$  on  $[0, 1]^\Gamma$ .

In Section 5 of [10] it is asked for which  $k$  can automorphism-invariant random  $k$ -colorings of Cayley graphs be attained as IID factors (see also [1, Question 10.5]). In [4, Corollary 6.4] translation-invariant random  $d$ -colorings of Cayley graphs are constructed, where as usual  $d$  is the degree of the graph, but this involves passing to actions weakly equivalent to the Bernoulli shift (or alternatively taking weak limits of IID factors). We can now strengthen this result, giving  $d$ -colorings as IID factors except in the cases  $\mathbb{Z}$  and  $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$  where there is an obvious ergodic-theoretic obstruction.

**Corollary 7.4.** *Suppose that  $\Gamma$  is a countable group not isomorphic to  $\mathbb{Z}$  or  $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ , and suppose that  $S$  is a finite symmetric generating set for  $\Gamma$  with  $|S| = d$ . Then there is an automorphism-invariant random  $d$ -coloring of  $\text{Cay}(\Gamma, S)$  which is an IID factor.*

*Proof.* In the case that  $\Gamma$  is amenable, it has finitely many ends, and so we can apply [4, Theorem 6.7]. Otherwise,  $\Gamma$  is nonamenable and we can apply Corollary 5.6 to obtain from  $\text{WMSF}(G'(\Gamma, S))$  a Borel  $d$ -coloring  $c: B \rightarrow d$  of the restriction of  $G(\Gamma, S)$  to the conull set  $B \subseteq [0, 1]^\Gamma$  on which  $\text{WMSF}(G'(\Gamma, S))$  has one end. Define  $\pi: B \rightarrow \text{Col}(\Gamma, S, d)$  by  $(\pi(x))(\gamma) = c(\gamma^{-1} \cdot x)$ . Then  $\pi_*\mu$  is a translation-invariant random  $d$ -coloring which is a factor of IID by construction, where as usual  $\pi_*\mu(A) = \mu(\pi^{-1}(A))$ .  $\square$

**Remark 7.5.** Russ Lyons (private communication) points out that this method of proof using spanning forests works for finitely generated groups of more than linear growth by using instead the wired *uniform* spanning forest (WUSF); see Section 10 of [3]. The realization of the WUSF as a factor of IID follows from Wilson's algorithm rooted at infinity (see [7, Proof of Proposition 9]) in the transient case and Pemantle's strong Følner independence [14] in the amenable case.

## 8. THE CASE $d = 2$

In this section, we prove Theorem 1.3, giving a measurable analogue of Brooks' theorem for the case  $d = 2$ .

Given a graph  $G$  on  $X$ , let the equivalence relation  $E_{2,G}$  be the equivalence relation on  $X$  where  $x E_{2,G} y$  if  $x$  and  $y$  are connected by a path of even length in  $G$ . Then in the case where  $X$  is finite, we can rephrase the existence of an odd cycle in the following way: there is a nonempty  $G$ -invariant subset  $A$  of  $X$  such that every nonempty  $E_{2,G}$ -invariant subset of  $A$  is  $G$ -invariant.

Now, in the measurable context, even without the presence of odd cycles, there are Borel graphs  $G$  and measures  $\mu$  for which every  $E_{2,G}$ -invariant Borel set differs by a nullset from a Borel  $G$ -invariant set. For example, the Borel graph  $G_S = \{(x, y) \in \mathbb{T}^2 : S(x) = y \vee S(y) = x\}$  induced by an irrational rotation  $S: \mathbb{T} \rightarrow \mathbb{T}$  of the unit circle is 2-regular and acyclic, and since  $S^2$  is ergodic with respect to Lebesgue measure, every non-null  $E_{2,G}$ -invariant Borel set is Lebesgue conull. It follows that  $G_S$  does not admit a  $\mu_{\mathbb{T}}$ -a.e. Borel 2-coloring, as the color sets in a measurable 2-coloring would have to be disjoint,  $E_{2,G}$ -invariant, and non-null since  $G_S$  is induced by a measure preserving transformation and is hence quasi-invariant. Likewise, there is no Baire measurable 2-coloring of  $G_S$  with respect to the usual topology on  $\mathbb{T}$  since every non-meager  $E_{2,G}$ -invariant Borel set of vertices in  $G_S$  is comeager.

If we regard the phenomenon described above as generalization of possessing an odd cycle, then we have the following generalization of Brooks's theorem in the case  $d = 2$ :

**Theorem 8.1.** *Suppose  $G$  is a Borel graph on a standard Borel space  $X$  with vertex degree bounded by  $d = 2$  such that  $G$  contains no odd cycles. Let  $E_{2,G}$  be the equivalence relation on  $X$  where  $x E_{2,G} y$  if  $x$  and  $y$  are connected by a path of even length in  $G$ .*

- (1) *Let  $\mu$  be a  $G$ -quasi-invariant Borel probability measure on  $X$ . Then  $G$  admits a  $\mu$ -measurable 2-coloring if and only if there does not exist a non-null  $G$ -invariant Borel set  $A$  such that every  $E_{2,G}$ -invariant Borel subset of  $A$  differs from a  $G$ -invariant set by a nullset.*
- (2) *Let  $\tau$  be a  $G$ -quasi-invariant Polish topology compatible with the Borel structure on  $X$ . Then  $G$  admits a Baire measurable 2-coloring if and only if there does not exist a non-meager  $G$ -invariant Borel set  $A$  so that every  $E_{2,G}$ -invariant Borel subset of  $A$  differs from a  $G$ -invariant set by a meager set.*

*Proof.* We prove just part 1, since the proof of 2 is similar. Assume first that  $G$  admits a  $\mu$ -measurable 2-coloring with colors sets  $C_0$  and  $C_1$ . Now  $C_0$  and  $C_1$  must both be non-null, since  $\mu$  is  $G$ -quasi-invariant. However, if  $A$  is a non-null  $G$ -invariant Borel set, then  $A \cap C_0$  is  $E_{2,G}$ -invariant, however  $A \cap C_0$  cannot differ from a  $G$ -invariant Borel set by a nullset since  $\mu$  is  $G$ -quasi-invariant.

For the converse, assume that for every  $\mu$ -measurable non-null  $G$ -invariant set  $A$  we can find a  $\mu$ -measurable  $E_{2,G}$ -invariant  $C \subseteq A$  which is not within a null-set of being  $G$ -invariant. Then the sets  $C_0 = \{x \in C : [x]_{E_G} \setminus C \neq \emptyset\}$  and  $C_1 = [C_0]_{E_G} \setminus C_0$  are non-null, and they determine a  $\mu$ -measurable 2-coloring of  $G \upharpoonright [C_0]_{E_G}$ . We may continue this process on  $X \setminus [C_0]_{E_G}$ , and by measure theoretic exhaustion we can obtain a  $\mu$ -measurable 2-coloring  $c : Y \rightarrow \{0, 1\}$  of  $G \upharpoonright Y$  for some  $G$ -invariant conull  $Y \subseteq X$ .  $\square$

#### A. BOREL VS MEASURABLE COLORINGS

Let  $(\mathbb{Z}/2\mathbb{Z})^{*d}$  be the  $d$ -fold free product of the group  $\mathbb{Z}/2\mathbb{Z}$ . This group acts via the left shift action on the standard Borel space  $\mathbb{N}^{(\mathbb{Z}/2\mathbb{Z})^{*d}}$ . Let  $X = \{x \in \mathbb{N}^{(\mathbb{Z}/2\mathbb{Z})^{*d}} : \gamma \cdot x \neq x \text{ for all nonidentity } \gamma \in (\mathbb{Z}/2\mathbb{Z})^{*d}\}$  be the free part of this action. Let  $G((\mathbb{Z}/2\mathbb{Z})^{*d}, \mathbb{N})$  be the Borel graph on  $X$  where  $x$  is adjacent to  $y$  if there is a generator  $\gamma$  of  $(\mathbb{Z}/2\mathbb{Z})^{*d}$  such that  $\gamma \cdot x = y$  or  $\gamma \cdot y = x$ . Note this graph is acyclic and  $d$ -regular. As discussed in the introduction, Marks [12, Theorem 1.2] has shown that this graph has no Borel  $d$ -coloring. However, our Theorem 1.2 shows that for every  $d \geq 3$ , there is a  $\mu$ -measurable and Baire measurable  $d$ -coloring of  $G((\mathbb{Z}/2\mathbb{Z})^{*d}, \mathbb{N})$  with respect to any Borel probability measure or compatible Polish topology on  $X$ .

Hence, for all finite  $d \geq 3$ , for Borel graphs  $G$ , admitting  $\mu$ -measurable  $d$ -coloring with respect to every Borel probability measure is a strictly weaker notion than admitting a Borel coloring, as is admitting a Baire measurable  $d$ -coloring with respect to every compatible Polish topology, as witnessed by these explicit graphs given above. In this appendix, we show that in the case

$d = 2$ , these notions are the same, even without any degree assumptions on  $G$ .

**Proposition A.1.** *Let  $G$  be a locally countable Borel graph on a standard Borel space  $X$ . Then the following are equivalent:*

- (1)  $G$  admits a Borel 2-coloring.
- (2) For every Borel probability measure  $\mu$  on  $X$ ,  $G$  admits a  $\mu$ -measurable 2-coloring.
- (3) For every compatible Polish topology  $\tau$  on  $X$ ,  $G$  admits a Baire measurable 2-coloring.

*Proof.* This is actually a Corollary of a more general unpublished result of Louveau (see [13, Theorem 15]), but we note that there is also a quick proof using the  $G_0$ -dichotomy [9, Theorem 6.6]. We will prove the equivalence of 1. and 2., since the proof of the equivalence of 1. and 3. is similar. It suffices to show that if  $G$  admits no Borel 2-coloring then  $G$  admits no  $\mu$ -measurable 2-coloring for some Borel probability measure  $\mu$  on  $X$ . If  $G$  contains an odd cycle then it cannot be 2-colored at all, so we may assume that  $G$  contains no odd cycles. Let  $G^{\text{odd}} = \{(x, y) \in X^2 : \text{dist}_G(x, y) \text{ is odd}\}$ , where  $\text{dist}_G : X^2 \rightarrow \mathbb{N} \cup \{\infty\}$  denotes the graph distance in  $G$ . Then  $G^{\text{odd}}$  admits no Borel  $\mathbb{N}$ -coloring. (Otherwise, by [9, Proposition 4.2] there is a maximal  $G^{\text{odd}}$ -independent set  $A \subseteq X$  which is Borel, and since  $G$  contains no odd cycles the set  $X \setminus A$  is  $G^{\text{odd}}$ -independent as well, which contradicts that  $G$  admits no Borel 2-coloring.) It follows from [9, Theorem 6.6] that there is an injective Borel homomorphism  $f : 2^{\mathbb{N}} \rightarrow X$  from the graph  $G_0$  to  $G^{\text{odd}}$ . Then  $f$  is a homomorphism from  $G_0^{\text{odd}} = \{(u, v) \in (2^{\mathbb{N}})^2 : \text{dist}_{G_0}(u, v) \text{ is odd}\} = \{(u, v) \in 2^{\mathbb{N}} : u \text{ and } v \text{ differ on an odd number of coordinates}\}$  to  $G^{\text{odd}}$ . Let  $\nu$  denote the uniform product measure on  $2^{\mathbb{N}}$ . Then every Borel  $G_0^{\text{odd}}$ -independent set is  $\nu$ -null (see [5, Example 3.7]), hence every Borel  $G^{\text{odd}}$ -independent set is  $f_*\nu$ -null. Fix by Theorem 2.2 a sequence  $(T_i)_{i \in \mathbb{N}}$  of Borel automorphisms such that  $E_G = \bigcup_i \text{graph}(T_i)$  and let  $\mu = \sum_i 2^{-i} (T_i)_* f_*\nu$ , so that  $\mu$  is a  $G$ -quasi-invariant probability measure with the same  $G$ -invariant null sets as  $f_*\nu$ . Suppose toward a contradiction that there is a  $\mu$ -measurable 2-coloring of  $G$ . Then there is a Borel 2-coloring  $c : B \rightarrow \{0, 1\}$  of  $G \upharpoonright B$  for some  $G$ -invariant  $\mu$ -conull Borel subset  $B \subseteq X$ . Since  $B$  is invariant,  $c$  is a 2-coloring of  $G^{\text{odd}} \upharpoonright B$ , and since  $B$  is  $\mu$ -conull it is  $f_*\nu$ -conull, so either  $c^{-1}(0)$  or  $c^{-1}(1)$  is a Borel  $G^{\text{odd}}$ -independent set with positive  $f_*\nu$ -measure, a contradiction.  $\square$

## B. QUASI-INVARIANT TOPOLOGIES

In this section, we prove the following proposition:

**Proposition B.1.** *Suppose that  $E$  is a countable Borel equivalence relation on a standard Borel space  $X$ , and  $\tau$  is a compatible Polish topology for  $X$ . Then there is an  $E$ -quasi-invariant compatible Polish topology  $\tau'$  for  $X$  such that every  $\tau'$ -meager set is  $\tau$ -meager.*

*Proof.* By the Feldman-Moore theorem, let  $\{T_i\}_{i \in \mathbb{N}}$  be countably many Borel automorphisms of  $X$  such that  $E = \bigcup_{i \in \mathbb{N}} \text{graph}(T_i)$ .

We claim there exists a comeager  $G_\delta$  set  $A$  and open sets  $\{U_i\}_{i \in \mathbb{N}}$  such that setting  $B_i = T_i(A \cap U_i)$ , we have that the sets  $A, B_0, B_1, \dots$  are pairwise disjoint, their union  $A \cup \bigcup_{i \in \mathbb{N}} B_i$  is  $E$ -invariant, and for every open  $U$ , the set  $A \cap T_i(A \cap U)$  is open in  $A$ . Given this claim we are finished, since we can define  $\tau'$  as follows. Let  $Y = A \cup \bigcup_{i \in \mathbb{N}} B_i$ . Make the sets  $A, B_0, B_1, \dots$  all open in  $\tau'$ . Let the subspace topology on  $A$  in  $\tau'$  be the same as in  $\tau$  (note that since  $A$  is  $G_\delta$ , this is Polish) and choose the subspace topology on  $B_i$  so that  $T_i \upharpoonright A \cap U_i$  is a homeomorphism in  $\tau'$ . Note now that  $T_i \upharpoonright Y$  is continuous for every  $Y$  by our condition above. Finish by defining  $\tau'$  on  $X \setminus Y$  to be a compatible Polish topology so that the  $T_i$  are continuous on all of  $X$ . Now since  $A$  is comeager in  $\tau$ , and open in  $\tau'$ , and the subspace topology on  $A$  is the same in  $\tau$  and  $\tau'$ , it is clear that every  $\tau$ -meager set is also  $\tau'$ -meager.  $\tau'$  is  $E$ -quasi-invariant because the  $T_i$  are continuous for  $\tau'$ .

We turn now to the construction of  $A$  and  $\{U_i\}_{i \in \mathbb{N}}$ , working from now on only in the topology  $\tau$ . We begin by recalling a few standard facts. Suppose  $T$  is a Borel automorphism of  $X$ . Consider the collection of open sets  $U$  such that there is a meager set  $M \subseteq X$  such that  $T^{-1}(M)$  is comeager in  $U$ . Clearly this collection of open sets is closed under unions, and hence there is a maximal such open set  $\hat{U}$ . Given this  $\hat{U}$ , note further, that if  $C$  is any comeager set and open  $U'$  is disjoint from  $\hat{U}$ , then  $T^{-1}(C)$  is comeager in  $U'$ , else  $T^{-1}(C)$  is meager inside some open  $U'' \subseteq U'$ , and so  $T(U'' \setminus T^{-1}(C))$  is meager (being disjoint from  $C$ ) and hence  $\hat{U}$  is not maximal.

We begin by defining a sequence of meager sets  $\{M_i\}_{i \in \mathbb{N}}$  and our desired  $\{U_i\}_{i \in \mathbb{N}}$  simultaneously. Given  $A_0, \dots, A_{n-1}$ , let  $U_n$  be the maximal open set such that there is a meager set  $M_n \subseteq X$  such that  $T^{-1}(M_n)$  is comeager inside  $U_n$ , and  $T_i^{-1}(A_i \cap M)$  is meager for every  $i < n$ . Note now that for every  $A \subseteq X \setminus M_0 \cup \dots \cup M_n$ , that  $A, T_0(U_0), \dots, T_n(U_n)$  are all disjoint. Now if  $A$  is any comeager subset of  $X \setminus \bigcup_{i \in \mathbb{N}} M_i$ , this already insures that the sets  $A, T(A \cap U_0), T(A \cap U_1), \dots$  are all disjoint. To finish, by taking a sufficiently comeager  $G_\delta$  set  $A$ , we can ensure that  $A \cap T_i(A \cap U)$  is open in  $A$  for every open  $U$ , and that  $A \cup \bigcup_{i \in \mathbb{N}} T_i(A \cap U_i)$  is  $E$ -invariant.  $\square$

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