Questions, Conjectures and Remarks on Globally Rigid Tensegrities

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Abstract

This is a quick review of properties of stress matrices with respect to the global rigidity of tensegrity frameworks, and recent results about generic global rigidity. Then there are some applications and connections to finding specific geometric configurations that are globally rigid. Several conjectures and questions are mentioned. Also, a proof is given that vertex splitting preserves generic and geometric global rigidity under a mild additional assumption on the starting framework.

1 Introduction

Most of the recent results concerning global rigidity have been concerned with generic global rigidity of bar frameworks. In [6], I showed that if a bar framework $G(\mathbf{p})$ has a stress matrix Ω of maximal rank and it is infinitesimally rigid, then it is globally rigid when the configuration $\mathbf{p}$ is generic. This result was motivated by a similar condition for a tensegrity, but with the additional condition that Ω be positive semi-definite. Then in [3, 17] Berg, Jackson, and Jordán show that in the plane, generic redundant rigidity and

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vertex 3-connectedness is equivalent to generic global rigidity. This gives a entirely combinatorial condition for generic global rigidity in the plane for bar frameworks, and this condition can be calculated deterministically in polynomial time. Meanwhile, in [11], Gortler, Healy, and Thurston show that my condition on the stress matrix Ω, is necessary as well as sufficient for generic global rigidity for bar frameworks in $\mathbb{E}^d$. (This is all assuming that the bar framework is not a simplex.)

One of the drawbacks of these results is that the generic hypothesis is ambiguous as to just what configurations are to be avoided to insure global rigidity. Even when $G(\mathbf{p})$ has a stress matrix Ω of maximal rank and it is infinitesimally rigid, it may not be globally rigid in $\mathbb{E}^d$ at the configuration $\mathbf{p}$ such as Figure (1a). This is in contrast to generic rigidity itself, because when $G(\mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^d$, it is automatically rigid.

Here I will discuss some conjectures, questions, and a few results where it may be possible to determine certain determined configurations $\mathbf{p}$ where it is possible to say that $G(\mathbf{p})$ is globally rigid at the configuration $\mathbf{p}$, which can be constructed fairly reasonably. A natural point of view for these constructions is to consider G to be a tensegrity with cables and struts, where cables have a positive stress and struts have a negative stress, instead of bars. The cables are only constrained to not increase in length, and the struts are constrained to not decrease in length.

A conceit I like is to say the tensegrity $G(\mathbf{p})$ *dominates* the tensegrity $G(\mathbf{q})$, and write $G(\mathbf{q}) \leq G(\mathbf{p})$, for two configurations $\mathbf{q}$ and $\mathbf{p}$, if

$$\begin{array}{rcll} |\mathbf{p}_i - \mathbf{p}_j| & \geq & |\mathbf{q}_i - \mathbf{q}_j| & \text{for } \{i,j\} \text{ a cable,} \\ |\mathbf{p}_i - \mathbf{p}_j| & \leq & |\mathbf{q}_i - \mathbf{q}_j| & \text{for } \{i,j\} \text{ a strut and} \\ |\mathbf{p}_i - \mathbf{p}_j| & = & |\mathbf{q}_i - \mathbf{q}_j| & \text{for } \{i,j\} \text{ a bar.} \end{array} \tag{1}$$

We say a tensegrity $G(\mathbf{p})$ is *globally rigid* in $\mathbb{E}^d$ if for any other configuration $\mathbf{q}$ of the same labeled nodes in $\mathbb{E}^d$, $G(\mathbf{q}) \leq G(\mathbf{p})$ implies that $\mathbf{q}$ is congruent to $\mathbf{p}$. In other words, if the member constraints of (1) are satisfied by $\mathbf{q}$, then there is a rigid congruence of $\mathbb{E}^d$ given by a d-by-d orthogonal matrix A and a vector $\mathbf{b} \in \mathbb{E}^d$ such that for $i = 1, \dots, n$, $\mathbf{q}_i = A\mathbf{p}_i + \mathbf{b}$. Indeed, even more strongly, regard $\mathbb{E}^d \subset \mathbb{E}^D$, for $d \leq D$. If, even though $G(\mathbf{p})$ is in E^d, it is true that $G(\mathbf{p})$ is globally rigid in $\mathbb{E}^D$, for all $D \geq d$, then we say $G(\mathbf{p})$ is *universally globally rigid*. The example in Figure (1a) is rigid in the plane, but not globally rigid in the plane, since it can fold around a diagonal. Figure (1b) is globally rigid in the plane but not universally globally rigid, since it is flexible in three-space. Figure (1c) is universally globally rigid. These are all bar frameworks.

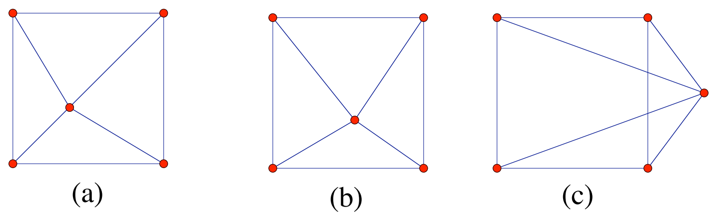

Figure 1: Three examples of planar rigid bar frameworks.

The local and global rigidity of the examples in Figure 1 are fairly easy to determine, but what are some tools to use for more complicated tensegrities?

The energy function E_ω helps. Let $\omega = (\dots, \omega_{ij}, \dots)$ be a *proper stress* for a tensegrity graph G (i.e $\omega_{ij} \geq 0$ for $\{i,j\}$ a cable, and $\omega_{ij} \leq 0$ for $\{i,j\}$ a strut). For any configuration $\mathbf{p}$ of nodes in $\mathbb{E}^d$, define the *stress-energy* associated to ω as

$$E_\omega(\mathbf{p}) = \sum_{i<j} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)^2, \tag{2}$$

where the product of vectors is the ordinary dot product, and the square of a vector is the square of its Euclidean length.

So if $G(\mathbf{p})$ dominates $G(\mathbf{q})$ and ω is a proper stress for G, then $E_\omega(\mathbf{p}) \geq E_\omega(\mathbf{q})$, and when ω is strict and $E_\omega(\mathbf{p}) = E_\omega(\mathbf{q})$, then $|\mathbf{p}_i - \mathbf{p}_j| = |\mathbf{q}_i - \mathbf{q}_j|$ for all the members $\{i,j\}$ of G. The conditions of (1) are called the *tensegrity constraints*.

The idea is to look for situations when the configuration $\mathbf{p}$ is a minimum for the functional E_ω. The first step is to determine when $\mathbf{p}$ is a critical point for E_ω. This will happen when all directional derivatives given by $\mathbf{p}' = (\mathbf{p}'_1, \dots, \mathbf{p}'_n)$ starting at $\mathbf{p}$ are 0. So we perform the following calculation starting from (2) for $0 \leq t \leq 1$:

$$E_\omega(\mathbf{p} + t\mathbf{p}') = \sum_{i<j} \omega_{ij}((\mathbf{p}_i - \mathbf{p}_j)^2 + 2t(\mathbf{p}_i - \mathbf{p}_j)(\mathbf{p}'_i - \mathbf{p}'_j) + t^2(\mathbf{p}'_i - \mathbf{p}'_j)^2).$$

Taking derivatives and evaluating at $t = 0$, we get:

$$\frac{d}{dt} E_\omega(\mathbf{p} + t\mathbf{p}')|_{t=0} = 2\sum_{i<j} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)(\mathbf{p}'_i - \mathbf{p}'_j). \tag{3}$$

At a critical configuration $\mathbf{p}$, equation (3) must hold for all directions $\mathbf{p}'$, so the following equilibrium vector equation must hold for each node i:

$$\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0. \tag{4}$$

When equation (4) holds for all $i = 1, \dots, n$, we say ω is an *equilibrium stress* or equivalently a *self stress*, or just a stress for $\mathbf{p}$ when the equilibrium is clear from the context. To get an understanding of how this works, consider the example of a square in the plane as in Figure 2. It is easy to see that the

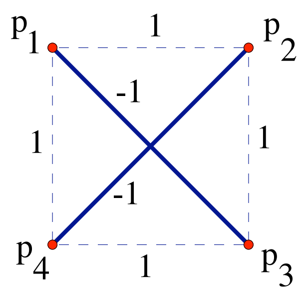

Figure 2: A square tensegrity with its diagonals, where a proper equilibrium stress is indicated.

vector equilibrium equation (4) holds for the three vectors at each node, even though many people tend to put $-\sqrt{2}$ instead of -1 for the strut stresses.

2 The stress matrix

The stress-energy function E_ω defined by (2) is really a quadratic form. It is an easy matter to compute the (symmetric) matrix associated to that quadratic form. For any stress ω, where $\omega_{ij} = \omega_{ji}$ for all $1 \le i \le j \le n$, define the associated n-by-n *stress matrix* Ω such that the (i, j) entry is $-\omega_{ij}$ for $i \neq j$, and the diagonal entries are such that the row and column sums are 0. Recall that any stress ω_{ij} not designated in the vector form $\omega = (\cdots, \omega_{ij}, \cdots)$ is assumed to be 0.

With this terminology regard a configuration $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ in $\mathbb{E}^d$ as a column vector. We see that E_ω is essentially given by the matrix Ω repeated

d times. The tensor product of matrices (or sometimes called the Kronecker product) gives the matrix of E_ω as $\Omega \otimes I^d$, and

$$E_\omega(\mathbf{p}) = (\mathbf{p})^T \Omega \otimes I^d \mathbf{p}.$$

It is also convenient to rewrite the equilibrium condition (4) in terms of matrices. Define the *configuration matrix* for the configuration $\mathbf{p}$ as

$$P = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

a $(d+1)$-by-n matrix, and the equilibrium condition (4) is equivalent to

$$P\Omega = \mathbf{0}.$$

Each coordinate of P as a row vector multiplied on the right by Ω represents the equilibrium condition in that coordinate. The last row of ones of P represent the condition that the column sums (and therefore the row sums) of Ω are $\mathbf{0}$. It is also easy to see that the linear rank of P is the same as the dimension of the affine span of $\mathbf{p}_1, \dots, \mathbf{p}_n$ in $\mathbb{E}^d$.

Suppose that we add rows to P until all the rows span the co-kernel of Ω. The corresponding configuration $\mathbf{p}$ will be called a *universal configuration* for ω (or equivalently Ω).

Proposition 1. *If $\mathbf{p}$ is a universal configuration for ω, any other configuration $\mathbf{q}$, which is in equilibrium with respect to ω, is an affine image of $\mathbf{p}$.*

Proof. Let Q be the configuration matrix for $\mathbf{q}$. Since the rows of P are a basis for the co-kernel of Ω, and the rows of Q are, by definition, in the co-kernel of Ω, there is a $(d+1)$-by-$(d+1)$ matrix A such that $AP = Q$. Since P and Q share the last row of ones, we know that A takes the form

$$A = \begin{pmatrix} A_0 & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix},$$

where A_0 is a d-by-d matrix, $\mathbf{b}$ is a 1-by-d matrix (a vector in $\mathbb{E}^d$), and the last row is all 0's except for the 1 in the lower right hand entry. Then we see that for each $i = 1, \dots, n$, $\mathbf{q}_i = A_0\mathbf{p}_i + \mathbf{b}$, as desired. □

The stress matrix plays a central role in what follows. Note that when the configuration $\mathbf{p}$ is universal, with affine span all of $\mathbb{E}^d$, for the stress ω, the dimension of the co-kernel (which is the dimension of the kernel) of Ω is d, and the rank of Ω is $n-d-1$. But even when the configuration $\mathbf{p}$ is not universal for ω, it is the projection of a universal configuration, and so the rank $\Omega \leq n-d-1$.

Projective transformations of the configuration $\mathbf{p}$ are also very friendly with regard to stress matrices.

Proposition 2. *If $\mathbf{p}$ is a configuration of n nodes in $\mathbb{E}^d$ with stress ω and stress matrix Ω, then any non-singular projective image $\mathbf{q}$ of $\mathbf{p}$ has a corresponding stress $\bar{\omega}$ with stress matrix given by $\bar{\Omega} = D\Omega D$, where D is an n-by-n diagonal matrix with non-zero diagonal entries.*

Proof. The argument is similar to Proposition 1. The projective transformation can be described as $AP = \tilde{Q}$, where P is the configuration matrix for $\mathbf{p}$, the columns of $\tilde{Q}$ represent the nodes of $\mathbf{q}$ up to scaling in $\mathbb{E}^{d+1}$, and A is a non-singular $(d+1)$-by-$(d+1)$ matrix such that each entry of the bottom row of $\tilde{Q}$ is non-zero. Then the configuration matrix for $\mathbf{q}$ is

$$Q = APD^{-1} = \tilde{Q}D^{-1} = \begin{pmatrix} \tilde{\mathbf{q}}_1 & \tilde{\mathbf{q}}_2 & \cdots & \tilde{\mathbf{q}}_n \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix} D^{-1} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

where D is the diagonal matrix with $\lambda_1, \lambda_2, \cdots, \lambda_n$ in the main diagonal. Then $Q\bar{\Omega} = QD\Omega D = APD^{-1}D\Omega D = AP\Omega D = \mathbf{0}$. Thus $\mathbf{q}$ is in equilibrium with respect to $\bar{\omega}$ as required. $\square$

Note that the sign of the stresses may change under a projective transformation, but the signature of the stress matrix (the number of positive eigenvalues, the number of negative eigenvalues, and the number of zero eigenvalues) does not change. Indeed, a cable becomes a strut, or a strut becomes a cable, when a point on its relative interior is sent to infinity. Figure 3 shows how this works for a tensegrity similar to Figure 5 in [19]. Figure (3a) is a Roth polygon as in [4].

3 The fundamental theorem

We come to one of the basic tools for showing specific tensegrities are globally rigid and more. If ω is a proper equilibrium stress for the tensegrity $G(\mathbf{p})$, and $\omega_{ij} \neq 0$, then $\mathbf{p}_i - \mathbf{p}_j$ is called a *stressed direction,* and the member

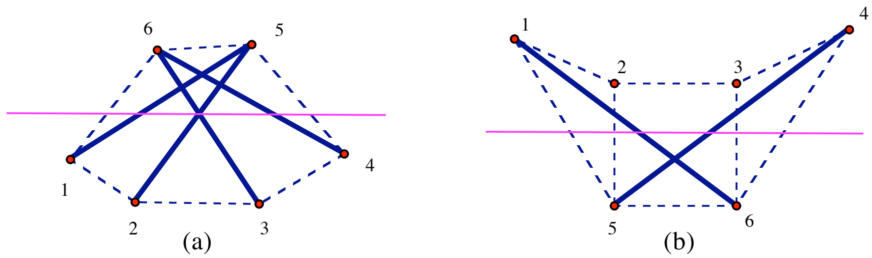

Figure 3: Figure (a) is transformed by a projective transformation to get Figure (b). The corresponding nodes are labeled and the line that is sent to infinity is shown. Each member that crosses that line has changed from a cable to a strut and vice-versa.

$\{i, j\}$ is called a *stressed member*. Note that if $G(\mathbf{q}) \leq G(\mathbf{p})$, $\omega_{ij} \neq 0$, and $|\mathbf{p}_i - \mathbf{p}_j| \neq |\mathbf{q}_i - \mathbf{q}_j|$, then $E_\omega(\mathbf{q}) < E_\omega(\mathbf{p})$. So if $\mathbf{p}$ is a configuration for the minimum of E_ω, the stressed members are effectively bars.

Theorem 3. *Let $G(\mathbf{p})$ be a tensegrity, where the affine span of $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is all of $\mathbb{E}^d$, with a proper equilibrium stress ω and stress matrix Ω. Suppose further*

1.) Ω is positive semi-definite.

2.) The configuration $\mathbf{p}$ is universal with respect to the stress ω. (In other words, the rank of Ω is $n - d - 1$.)

3.) The underlying bar framework $\bar{G}(\mathbf{p})$ is rigid, where $\bar{G}$ replaces each member $\{i, j\}$ with $\omega_{ij} \neq 0$ of G with a bar.

Then $G(\mathbf{p})$ is universally globally rigid.

The proof of this is in [4]. Condition 3.) can be replaced with the condition that the stressed directions, regarded as points in the projective space $\mathbb{RP}^{d+1}$, do not lie on a quadric at infinity. Without Condition 3.) there could be an affine flex of the configuration, but that is the only possibility. So any condition, such as infinitesimal rigidity, that insures rigidity is enough for Condition 3.), but infinitesimal rigidity is not necessary.

With this in mind, we say that a tensegrity is *super stable* if it has a proper equilibrium stress ω such that Conditions 1.), 2.) and 3.) hold. If just Conditions 1.) and 3.) hold and ω is strict (all members stressed), then we say $G(\mathbf{p})$ is *unyielding*. An unyielding tensegrity, essentially, has all its members replaced by bars.

Figure (1a) and Figure (1b) have Conditions 2.) and 3.) hold, but not Condition 1.), while Figure (1b) is globally rigid in the plane and Figure (1a) is not even globally rigid in the plane. Neither of these frameworks are universally globally rigid. There is only a one-dimensional stress space and the corresponding stress matrices (for non-zero stresses) have eigenvalues of opposite sign. For Figure (1c), on the other hand, there is a stress matrix where the two non-zero eigenvalues are positive. This is a Cauchy polygon of [4] which super stable as a tensegrity. Similarly both tensegrities of Figure 3 are also super stable, because, for example, Figure (3a) is super stable by [4].

In order to be super stable, there must be a single stress that stabilizes the tensegrity and similarly for a bar framework. Can the situation be more complicated? This suggests the following question.

Question 1. *Is there bar framework $G(\mathbf{p})$ in $\mathbb{E}^d$ (necessarily with a stress space of dimension at least 2) such that no equilibrium stress is positive semi-definite with rank $n-d-1$, but it is such that for any $\mathbf{p}'$ not an affine image of $\mathbf{p}$, there is an equilibrium stress with stress matrix Ω such that $(\mathbf{p}')^T\Omega \otimes I^d\mathbf{p}' > 0$?*

There are linear spaces of quadratic forms such that none are positive definite, but for every vector, there is some quadratic form in the linear space that is positive on that vector. Question 1 asks whether this is possible for stress matrices modulo the kernel corresponding to affine motions.

This is to be contrasted to a recent result of Gortler and Thurston [12].

Theorem 4. *If $G(\mathbf{p})$ is bar framework in $\mathbb{E}^d$, with $\mathbf{p}$ generic, and $G(\mathbf{p})$ universally globally rigid, then $G(\mathbf{p})$ is super stable. In other words there is a stress ω and corresponding stress matrix Ω, such that it has rank $n-d-1$ and is positive semidefinite. (Affine motions are easy to eliminate.)*

4 Affine transformations

An *affine transformation* or *affine map* of $\mathbb{E}^d$ is determined by a d-by-d matrix A and a vector $\mathbf{b} \in \mathbb{E}^d$. If $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ is any configuration in $\mathbb{E}^d$, an affine image is given by $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$, where $\mathbf{q}_i = A\mathbf{p}_i + \mathbf{b}$.

If the configuration $\mathbf{p}$ is in equilibrium with respect to the stress ω, then so is any affine transformation $\mathbf{q}$ of $\mathbf{p}$, as is seen by the following calculation:

$$\sum_j \omega_{ij}(\mathbf{q}_j - \mathbf{q}_i) = \sum_j \omega_{ij}(A\mathbf{p}_j + \mathbf{b} - A\mathbf{p}_i - \mathbf{b}) = A\sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0.$$

So our stress-energy functional E_ω can't "see" affine transformations, at least at critical points. Of course we know that when something is globally rigid, it cannot exclude rigid congruences, but the group of affine transformations are more than we would like. Notice that even projections, which are singular affine transformations, also preserve equilibrium configurations. Indeed, the equilibrium formula (4) is true if and only if it is true for each coordinate, which is the same as being true for orthogonal projections onto each coordinate axis.

This brings us to the question, for a tensegrity $G(\mathbf{p})$ in $\mathbb{E}^d$, when is there an affine transformation that preserves the member constraints (1)? It is clear that the matrix A is the only relevant part. For us, it will turn out that we also only need to consider when the members are bars. If $\{i, j\}$ is a bar of G, then the matrix A determines a transformation that preserves that bar length if and only if the following holds:

$$\begin{aligned} (\mathbf{p}_i - \mathbf{p}_j)^2 &= (\mathbf{q}_i - \mathbf{q}_j)^2 \\ &= (A\mathbf{p}_i - A\mathbf{p}_j)^2 \\ &= [A(\mathbf{p}_i - \mathbf{p}_j)]^T A(\mathbf{p}_i - \mathbf{p}_j) \\ &= (\mathbf{p}_i - \mathbf{p}_j)^T A^T A(\mathbf{p}_i - \mathbf{p}_j), \end{aligned}$$

or equivalently,

$$(\mathbf{p}_i - \mathbf{p}_j)^T(A^T A - I^d)(\mathbf{p}_i - \mathbf{p}_j) = 0 \tag{5}$$

where $()^T$ is the transpose operation, I^d is the d-by-d identity matrix, and vectors are regarded as column vectors in this calculation. If Equation (5) holds for all bars in $G(\mathbf{p})$, we say it has a *bar preserving affine image*, which is non-trival if A is not orthogonal. Similarly, $G(\mathbf{p})$ has a *non-trivial affine flex* if there is a continuous family of d-by-d matrices A_t, where $A_0 = I^d$, for t in some interval containing 0 such that each A_t satisfies Equation (5) for t in the interval.

This suggests the following definition. If $\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a collection of vectors in $\mathbb{E}^d$, we say that they lie on a *quadric at infinity* if there is a non-zero symmetric d-by-d matrix Q such that for all $\mathbf{v}_i \in \mathbf{v}$

$$\mathbf{v}_i^T Q \mathbf{v}_i = 0. \tag{6}$$

The reason for this terminology is that real projective space $\mathbb{RP}^{d-1}$ can be regarded as the set of lines through the origin in $\mathbb{E}^d$, and equation (6) is the definition of a quadric in $\mathbb{RP}^{d-1}$.

Notice that since the definition of an orthogonal matrix A is that $A^TA - I^d = \mathbf{0}$, the affine transformation defines a quadric at infinity if and only if the affine transformation is not a congruence.

Call the *bar directions of a bar tensegrity* the set $\{\mathbf{p}_i - \mathbf{p}_j\}$, for $\{i, j\}$ a bar of G. With this terminology, Equation (6) says that if the member directions of a bar tensegrity under an affine transformation A satisfy (5), they lie on a quadric at infinity. Conversely suppose that the member directions of a bar tensegrity $G(\mathbf{p})$ lie on a quadric at infinity in $\mathbb{E}^d$ given by a non-zero symmetric matrix Q. By the spectral theorem for symmetric matrices, we know that there is an orthogonal d-by-d matrix $X = (X^T)^{-1}$ such that:

$$X^TQX = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_d \end{pmatrix}.$$

Let λ_- be the smallest λ_i, and let λ_+ be the largest λ_i. Note $\infty \leq 1/\lambda_- < 1/\lambda_+ \leq \infty$, λ_- is non-positive, and λ_+ is non-negative when Q defines a non-empty quadric and when $1/\lambda_- \leq t \leq 1/\lambda_+$, $1 - t\lambda_i \geq 0$ for all $i = 1, \dots, d$. Working Equation (5) backwards for $1/\lambda_- \leq t \leq 1/\lambda_+$ we define:

$$A_t = X^T \begin{pmatrix} \sqrt{1-t\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{1-t\lambda_2} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{1-t\lambda_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{1-t\lambda_d} \end{pmatrix} X. \quad (7)$$

Substituting A_t from Equation (7) into Equation (5), we see that it provides a non-trivial affine flex of $G(\mathbf{p})$. If the configuration is contained in a lower dimensional hyperplane, we should really restrict to that hyperplane since there are non-orthogonal affine transformations that are rigid when restricted to the configuration itself. We have shown the following:

Proposition 5. *If $G(\mathbf{p})$ is a bar framework in $\mathbb{E}^d$, such that the nodes do not lie in a $(d-1)$-dimensional hyperplane, then it has a non-trivial bar preserving* *affine image if and only if it has a non-trivial bar preserving affine flex if and only if the bar directions lie on a quadric at infinity.*

Proposition 5 can be used to strengthen Theorem 3 by replacing Condition (3. with the condition that the stressed member directions do not lie on a quadric at infinity.

5 Generic global rigidity

It turns out that the problem of determining even when a bar framework is globally rigid is equivalent to a long list of problems known to be hard. See [22] for example. The problem of whether a cyclic chain of edges in the line has another realization with the same bar lengths, is equivalent to the uniqueness of a solution of the knapsack problem. This is one of the many problems on the list of NP complete problems.

One way to avoid this difficulty, is to assume that the configuration's coordinates are *generic*. This means that the coordinates of $\mathbf{p}$ in $\mathbb{E}^d$ are *algebraically independent over the rational numbers*, which means that there is no non-zero polynomial with rational coordinates satisfied by the coordinates of $\mathbf{p}$. This implies, among other things, that no $d+2$ nodes lie in a hyperplane, for example, and a lot more. In [6] I proved the following:

Theorem 6. *If $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ in $\mathbb{E}^d$ is generic and $G(\mathbf{p})$ is a rigid bar tensegrity in $\mathbb{E}^d$ with a non-zero stress matrix Ω of rank $n-d-1$, then $G(\mathbf{p})$ is globally rigid in $\mathbb{E}^d$.*

Notice that the hypothesis includes Conditions 2.) and 3.) of Theorem 3. The idea of the proof is to show that since the configuration $\mathbf{p}$ is generic, if $G(\mathbf{q})$ has the same bar lengths as $G(\mathbf{p})$, then they should have the same stresses. Then Proposition 1 applies.

Then recently in [11] S. Gortler, A. Healy, and D. Thurston proved the converse of Theorem 6 as follows:

Theorem 7. *If $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ in $\mathbb{E}^d$ is generic and $G(\mathbf{p})$ is a globally rigid bar tensegrity in $\mathbb{E}^d$, then either $G(\mathbf{p})$ is a bar simplex or there is stress matrix Ω for $G(\mathbf{p})$ with rank $n-d-1$.*

The idea here, very roughly, is to show that a map from an appropriate quotient of an appropriate portion of the space of all configurations has even topological degree when mapped into the space of edge lengths.

As pointed out in [11], using these results it is possible to find a polynomial time numerical (probablistic) algorithm that calculates whether a given graph is generically globally rigid in $\mathbb{E}^d$, and that the property of being globally rigid is a generic property. In other words, if $G(\mathbf{p})$ is globally rigid in $\mathbb{E}^d$ at one generic configuration $\mathbf{p}$, it is globally rigid at all generic configurations.

Interestingly, it is also shown in [11] that if $\mathbf{p}$ is generic in $\mathbb{E}^d$, and $G(\mathbf{q})$ has the same bar lengths in $G(\mathbf{p})$ in $\mathbb{E}^d$, then $G(\mathbf{p})$ can be flexed to $G(\mathbf{q})$ in $\mathbb{E}^{d+1}$.

A bar graph G is defined to be *generically redundantly rigid* in $\mathbb{E}^d$ if $G(\mathbf{p})$ is rigid at a generic configuration $\mathbf{p}$, and it remains rigid after the removal of any bar. A graph is vertex k-connected if it takes the removal of at least k vertices to disconnect the rest of the vertices of G. The following theorem of Hendrickson [13], provides two necessary conditions for generic global rigidity.

Theorem 8. *If $\mathbf{p}$ is a generic configuration in $\mathbb{E}^d$, and the bar tensegrity $G(\mathbf{p})$ is globally rigid in $\mathbb{E}^d$, then*

a.) G is vertex $(d+1)$-connected, and

b.) $G(\mathbf{p})$ is redundantly rigid in $\mathbb{E}^d$.

Condition a.) on vertex connectivity is clear since otherwise it is possible to reflect one component of G about the hyperplane determined by some d or fewer vertices. Condition b.) on redundant rigidity is natural since if, after a bar $\{\mathbf{p}_i, \mathbf{p}_j\}$ is removed, $G(\mathbf{p})$ is flexible, one watches as the distance between $\mathbf{p}_i$ and $\mathbf{p}_j$ changes during the flex, and waits until the distance comes back to it original length. If $\mathbf{p}$ is generic to start with, the new configuration will be not congruent to the original configuration.

Hendrickson conjectured that Conditions a.) and b.) were also sufficient for generic global rigidity, but it turns out in [5] that the complete bipartite graph $K_{5,5}$ in $\mathbb{E}^3$ is a counterexample.

The following is an application of some combinatorial results mentioned here that can by used to get specific geometric results about global rigidity. This was suggested by Konstatine Rybnikov.

Proposition 9. *Let $G(\mathbf{p})$ be a strictly convex triangulated sphere in $\mathbb{E}^3$, not a simplex. Then there is another configuration $\mathbf{q}$ in $\mathbb{E}^3$ such that $G(\mathbf{q})$ has the same edge lengths as $G(\mathbf{p})$.*

Proof. By Theorem 7 or Theorem 8 if $\mathbf{p}$ is generic in $\mathbb{E}^3$, then such a configuration q exists in $\mathbb{E}^3$ since the only equilibrium stress is $\mathbf{0}$. If $\mathbf{p}$ is not generic, there is a sequence of generic strictly convex triangulations converging to $\mathbf{p}$ and other corresponding configurations necessarily not convex. By pinning a vertex of $\mathbf{p}$, say, we can essentially restrict our space of triangulations to a compact subset of the configuration space. So there is a limiting configuration that is outside the space of strictly convex realizations (by Cauchy's Theorem or Dehn's Theorem about convex polyhedra) that has the same edge lengths as $G(\mathbf{p})$.

6 Dimensional rigidity

For a given bar framework $G(\mathbf{p})$ in $\mathbb{E}^d$ Alfakih in [2] defines it to be *dimensionally rigid* if any other framework $G(\mathbf{q})$ in $\mathbb{E}^N$ for $N \geq d$ with corresponding bar lengths the same, the configuration $\mathbf{q}$ lies in a d-dimensional affine subspace of $\mathbb{E}^N$. So if $G(\mathbf{p})$ is super stable in $\mathbb{E}^d$, for example, it is dimensionally rigid in $\mathbb{E}^d$. The following is in [2]. The proof below is direct in the sense that it gives the flex in $\mathbb{E}^d$ if the framework is not rigid, but still dimensionally rigid.

Theorem 10. *If a bar framework $G(\mathbf{p})$ is rigid in $\mathbb{E}^d$ and dimensionally rigid in $\mathbb{E}^d$, then it is* uniformally globally rigid. *That is $G(\mathbf{p})$ is globally rigid in $\mathbb{E}^N$ for all $N \geq d$.*

Proof. Let $G(\mathbf{q})$ be in $\mathbb{E}^N$ for $N \geq d$ with corresponding bar lengths the same as $G(\mathbf{p})$. In $\mathbb{E}^d \times \mathbb{E}^N = \mathbb{E}^{d+N}$ place the configuration $\mathbf{p}$ in $\mathbb{E}^d \times \mathbf{0}$, and place the configuration $\mathbf{q}$ in $\mathbf{0} \times \mathbb{E}^N$. For $0 \leq t \leq 1$ consider $\mathbf{p}_i(t) = (\cos(\pi t/2)\mathbf{p}_i, \sin(\pi t/2)\mathbf{q}_i)$ in $\mathbb{E}^d \times \mathbb{E}^N$ for each node i of G. So $\mathbf{p}(0)$ is $\mathbf{p} \times \mathbf{0}$, and $\mathbf{p}(1)$ is $\mathbf{0} \times \mathbf{q}$. It is easy to check that for each bar $\{i, j\}$ of G, $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ is constant for $0 \leq t \leq 1$. In other words, the corresponding bar lengths of $G(\mathbf{p}(t))$ are the same as $G(\mathbf{p})$, and thus by the dimensional rigidity assumption, each $G(\mathbf{p}(t))$ lies in a d-dimensional affine subspace of $\mathbb{E}^{N+d}$ for $0 \leq t \leq 1$. One can regard the affine subspace as fixed and $G(\mathbf{p}(t))$ lying in it. Then the rigidity hypothesis implies that all the $\mathbf{p}(t)$ for $0 \leq t \leq 1$ are congruent to $\mathbf{p}(0)$ and $\mathbf{p}$. In particular $\mathbf{p}(1)$ and $\mathbf{q}$ are congruent to $\mathbf{p}$, as desired. □

This can be generalized further, and in order to do this we consider some ways of understanding affine maps. Consider a function $f : X \to Y$. The *graph of f* is $\Gamma(f) = \{(x,y) \in X \times Y \mid f(x) = y\}$.

Lemma 11. *For any set $X \subset \mathbb{E}^d$, a function $f : X \to \mathbb{E}^N$, extends to an affine map $\hat{f} : \mathbb{E}^d \to \mathbb{E}^N$ if and only if $\Gamma(f)$ is contained in a d-dimensional affine subspace of $\mathbb{E}^d \times \mathbb{E}^N = \mathbb{E}^{d+N}$.*

Proof. Without loss of generality we assume that the affine span of X is all of $\mathbb{E}^d$. If $\hat{f}$ is an affine map extending f, then $\Gamma(\hat{f}) \subset \mathbb{E}^{d+N}$ is the d-dimensional subspace that is required. Conversely, if L is a d-dimensional affine subspace of $\mathbb{E}^{d+N}$ containing $\Gamma(f)$, its projection to $\mathbb{E}^d$ is an affine linear isomorphism. So the inverse of this projection followed by the projection into $\mathbb{E}^N$ is the required extension $\hat{f}$. □

The following is also in [2], but stated a bit differently.

Theorem 12. *If a framework $G(\mathbf{p})$ in $\mathbb{E}^d$ is dimensionally rigid and every affine image in $\mathbb{E}^d$ is a congruence, then $G(\mathbf{p})$ is universally globally rigid.*

Proof. Without loss of generality we assume that the affine span of $\mathbf{p}$ is all of $\mathbb{E}^d$. Let $G(\mathbf{q})$ be in $\mathbb{E}^N$ for $N \geq d$ with corresponding bar lengths the same as $G(\mathbf{p})$, and consider the same flex $\mathbf{p}_i(t) = (\cos(\pi t/2)\mathbf{p}_i, \sin(\pi t/2)\mathbf{q}_i)$ in $\mathbb{E}^d \times \mathbb{E}^N$ for each node i of G, as in the proof of Theorem 10. By dimensional rigidity $\mathbf{p}_i(\frac{1}{2}) = \frac{\sqrt{2}}{2}(\mathbf{p}_i, \mathbf{q}_i)$, for each node i of G, is contained in a d-dimensional affine linear subspace of $\mathbb{E}^{d+N}$. So $(\mathbf{p}_i, \mathbf{q}_i)$ is also contained in a d-dimensional linear subspace of $\mathbb{E}^{d+N}$. By Lemma 11, the configuration $\mathbf{q}$ is an affine image of $\mathbf{p}$. Thus $\mathbf{q}$ is congruent to $\mathbf{p}$. □

Using Propostion 5 we can test whether any bar framework $G(\mathbf{p})$ in $\mathbb{E}^d$ has an affine image that is not a congruence. We determine whether the bar directions lie on a quadric at infinity, which is simply a question of whether a set of linear equations has a non-zero solution.

Figure 4 is an example of a bar framework $G(\mathbf{p})$ in $\mathbb{E}^2$ that is dimensionally rigid but is a finite mechanism (necessarily in the plane). Note that there are exactly 2 bar directions, and thus lie on a quadric (two points) on the projective line at infinity. The affine flex is clear.

Each horizontal bar starting from $\mathbf{p}_1$ and $\mathbf{p}_2$ in Figure 4 is designated by $t_1\mathbf{p}_1'$, $t_2\mathbf{p}_1'$, $t_1\mathbf{p}_2'$, and $t_2\mathbf{p}_2'$, where t_1 and t_2 are non-zero distinct real numbers. So the vertical bar constraints are that $|\mathbf{p}_1 - \mathbf{p}_2| = |\mathbf{p}_1 + t\mathbf{p}_1' - (\mathbf{p}_2 + t\mathbf{p}_2')|$ for $t = 0, t_1, t_2$. Squaring both sides canceling and dividing by t we get the linear equation $-2(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{p}_1' - \mathbf{p}_2') + (\mathbf{p}_1' - \mathbf{p}_2')^2 t = 0$ for $t = t_1$ and $t = t_2$. Thus the linear equation has all 0 coefficients, and $\mathbf{p}_1' = \mathbf{p}_2'$. Thus the configuration is planar, and thus it dimensionally rigid but not rigid.

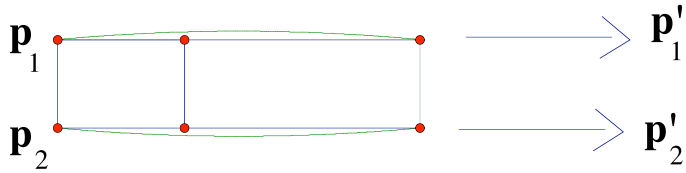

Figure 4: This bar framework has 6 nodes and 9 bars. The long horizontal bars are all parallel and are slightly curved to show their existence.

In [1] there is another version of Theorem 12 where it is assumed that the configuration is generic in $\mathbb{E}^d$ instead of directly assuming that it has no non-congruent affine flex. This depends on the following result in [6].

Theorem 13. *Suppose that $G(\mathbf{p})$ is framework in $\mathbb{E}^d$ such that each vertex of G has degree at least d, and the configuration $\mathbf{p}$ is generic. Then the member directions of $G(\mathbf{p})$ do not lie on a quadric at infinity.*

With this the following result in [1] is immediate.

Theorem 14. *Suppose that $G(\mathbf{p})$ is framework in $\mathbb{E}^d$ such that the configuration $\mathbf{p}$ is generic, the number of nodes $n \geq d+2$, and $G(\mathbf{p})$ is dimensionally rigid in $\mathbb{E}^d$. Then $G(\mathbf{p})$ is uniformally globally rigid.*

Proof. If any vertex $\mathbf{p}_i$ has degree less than d, since the configuration is generic, the $d+1$ other vertices have a d-dimensional affine span. Then rotating $\mathbf{p}_i$ into $\mathbb{E}^{d+1}$ creates anotheer configuration, whose affine span is $(d+1)$-dimensional with the same member lengths. If no vertex has degree less than d, then Theorem 13 applies with Theorem 12 applies to show that $G(\mathbf{p})$ is universally globally rigid. $\square$

The results in this section follow the same general outline of those in [2] and [1], but they instead rely on using the Gale diagram associated to a stress, which was not used here.

A configuration $\mathbf{p}$ in $\mathbb{E}^d$ is in *general position* if for $k \leq d+1$, the affine span of every subset of k points is $(k-1)$-dimensional. If a configuration $\mathbf{p}$ is generic, it is automatically in general position, but necessarily conversely. The next question asks whether we can get away with general position instead of generic in Theorem 14.

Question 2. *When the dimension $d \geq 3$, can the hypothesis that $\mathbf{p}$ is generic in Theorem 14 be replaced with $\mathbf{p}$ is in general position, with the same conclusion?*

The answer would be yes, if for a graph in general position such that the degree of each vertex is at least d is such that the member directions do not lie on a quadric at infinity. In the plane, the only connected framework, with each vertex of degree greater than or equal to 2 with the configuration in general position, is a single cycle, with each bar alternately lying in one and then the other direction, as in Figure 5.

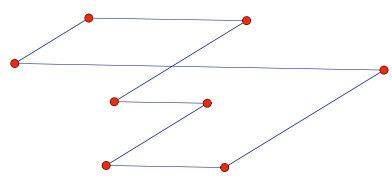

Figure 5: Each vertex is of degree 2, and each bar is parallel to one of two directions, which is a conic in the real projective line, while this configuration is in general position.

7 Infinitesimal rigidity

Before the discussion of the situation in the plane, it is helpful to recall some of the basic facts with regard to infinitesimal rigidity. Given a configuration $\mathbf{p}$ in $\mathbb{E}^d$, the *rigidity matrix* is defined as m-by-dn matrix

$$R(\mathbf{p}) = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \mathbf{p}_i - \mathbf{p}_j & 0 & \cdots & 0 & \mathbf{p}_j - \mathbf{p}_i & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where the entries in the d columns corresponding to node i and node j and row $\{i, j\}$ have $\mathbf{p}_i - \mathbf{p}_j$ and $\mathbf{p}_j - \mathbf{p}_i$ respectively (regarded as row vectors). All other entries are 0. The rows correspond to the m members of G. We say that a bar framework $G(\mathbf{p})$ with n nodes is *infinitesimally rigid* in $\mathbb{E}^d$ if it is either a simplex (where the configuration consists of $k \leq d + 1$ affine independent points, and G is the complete graph with all nodes connected by a bar) or $R(\mathbf{p})$ has rank $n - d(d+1)/2$, which turns out is maximal. The following is a basic result which is explained in [8].

Theorem 15. *If $G(\mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^d$, then it is rigid in $\mathbb{E}^d$.*

An easy calculation shows that a stress ω, regarded as a row vector, for a graph G is an equilibrium stress for $G(\mathbf{p})$ if and only if $\omega R(\mathbf{p}) = \mathbf{0}$. In other words the equilibrium stresses for $G(\mathbf{p})$ are the co-kernel of the rigidity matrix $R(\mathbf{p})$. So equivalently, $G(\mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^d$ if the dimension of the space of equilibrium stresses for $G(\mathbf{p})$ is $m - dn + d(d+1)/2$. It is useful to note that if $G(\mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^d$, then the rank of $R(\mathbf{p})$ is constant in an open neighborhood of the configuration $\mathbf{p}$ in $\mathbb{E}^{dn}$, and as $\mathbf{p}$ varies continuously in this neighborhood, the vector space of equilibrium stresses also varies continuously. So if a particular equilibrium stress ω is chosen for $G(\mathbf{p})$, an extension of ω can be chosen to vary continuously as well. In particular, the eigenvalues of the corresponding stress matrix for Ω will also vary continuously.

With these remarks, we get the following corollary of Theorem 3.

Corollary 16. *If a tensegrity $G(\mathbf{p})$ has a proper equilibrium stress, such that it is super stable and infinitesimally rigid as a bar graph, then G is generically globally rigid in $\mathbb{E}^d$.*

8 Edge splitting

In constructing globally rigid tensegrities, it is helpful to have operations that preserve global rigidity. One such is edge splitting, also called a Henneberg move, which is explained next, although it starts out restricting to only bar frameworks.

Start with an infinitesimally rigid bar framework $G(\mathbf{p})$ in $\mathbb{E}^d$, a bar $\{i, j\}$ of G, and $d - 1$ nodes $\mathbf{p}_1, \dots, \mathbf{p}_{d-1}$ such that the lines through $\mathbf{p}_i$ and $\mathbf{p}_j$ and $\mathbf{p}_1, \dots, \mathbf{p}_{d-1}$ are affine independent. Choose a new node $\mathbf{p}_0$ on the line through $\mathbf{p}_i$ and $\mathbf{p}_j$ but not equal to $\mathbf{p}_i$ or $\mathbf{p}_j$. Create a new bar graph G_1 by adding one extra node 0, deleting the bar $\{i, j\}$ and defining new bars $\{i, 0\}$, $\{0, j\}$, and $\{0, 1\}, \{0, 2\}, \dots, \{0, d-1\}$. Figure 6 shows this in the plane.

The following is a consequence of Theorem 6 and Theorem 7.

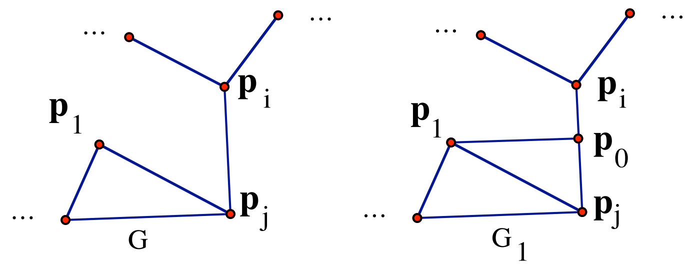

Figure 6: The bar framework $G(\mathbf{p})$ is split along the bar $\{i, j\}$ to get the framework $G_1(\mathbf{p})$.

Theorem 17. *If a bar graph G is generically globally rigid in $\mathbb{E}^d$, then so is G_1, which is obtained by splitting an edge of G.*

Proof. Let $\mathbf{p}$ be a generic configuration in $\mathbb{E}^d$. Then $G(\mathbf{p})$ is infinitesimally rigid. And as described above, $G_1(\hat{\mathbf{p}})$ is also infinitesimally rigid in $\mathbb{E}^d$ as well, where $\hat{\mathbf{p}}$ is the configuration $\mathbf{p}$ with the added nodes. This is easy to see by seeing that the dimension of the stress space for $G(\mathbf{p})$ and $G_1(\hat{\mathbf{p}})$ is the same since the number of members has increased by d and the number of nodes has increased by 1. The stress on the members $\{0, 1\}, \{0, 2\}, \dots, \{0, d-1\}$ is zero. So $G_1(\hat{\mathbf{p}})$ is infinitesimally rigid.

By Theorem 7, there is an (equilibrium) stress ω for $G(\mathbf{p})$ with a stress matrix Ω that has rank $n - d - 1$. The stress $\hat{\omega}$ for $G_1(\hat{\mathbf{p}})$ is the same as ω except that the stress ω_{ij} is replaced with the two stresses $\omega_{i0} = \pm\omega_{ij}|\mathbf{p}_i - \mathbf{p}_j|/|\mathbf{p}_i - \mathbf{p}_0|$ and $\omega_{j0} = \pm\omega_{ij}|\mathbf{p}_i - \mathbf{p}_j|/|\mathbf{p}_j - \mathbf{p}_0|$, where the sign depends on which part of the line through $\mathbf{p}_i$ and $\mathbf{p}_j$ that $\mathbf{p}_0$ is chosen. So in any universal configuration $\mathbf{p}_i$, $\mathbf{p}_j$ and $\mathbf{p}_0$ are collinear. Thus the affine span of any such configuration for $\hat{\mathbf{p}}$ would be the same as one for $\mathbf{p}$. Thus the dimension of kernel of the stress matrix $\hat{\Omega}$ for $\hat{\omega}$ is the same dimension as the kernel for Ω. By the remarks at the end of Section 7, the rank of $\hat{\Omega}$ and the rank of the rigidity matrix $R(\hat{\mathbf{p}})$ are unchanged by a small perturbation to a generic configuration. Thus G_1 is generically globally rigid in $\mathbb{E}^d$. □

I conjectured the following result of Bill Jackson and Tibor Jordán [15]. See also [3] for an earlier version when the graph is minimally redundantly rigid.

Theorem 18. *If G is vertex 3-connected and generically redundantly rigid bar graph in $\mathbb{E}^2$ (conditions a.) and b.) of Theorem 8), then starting with the* *complete graph K_4, one can find a sequence of graphs $K_4, G_1, \dots, G$, where each graph is obtained from the preceding one by an edge split or the insertion of an additional edge.*

Theorem 18 is a purely combinatorial result, but we obtain Hendrickson's conjecture in the plane by applying Theorem 17.

Corollary 19. *If G is vertex 3-connected and generically redundantly rigid bar graph in $\mathbb{E}^2$, then it is generically globally rigid in $\mathbb{E}^2$.*

9 Combinatorial versus geometric rigidity

The results of Theorem 18 and Corollary 19 are very nice, but there are some annoying aspects of such a purely combinatorial result. For a bar framework or tensegrity G, it is possible to find a configuration $\mathbf{p}$ and by calculating the rank of the rigidity matrix $R(\mathbf{p})$ explicitly determine that $G(\mathbf{p})$ is (infinitesimally) rigid in $\mathbb{E}^d$. Also there are classes of geometric configurations, such as triangulated strictly convex polyhedra in $\mathbb{E}^3$ (see [14], and [23] for not quite convex polyhedra) or pointed pseudo-triangulations in the plane (see [21]) that can be determined to be infinitesimally rigid.

For global rigidity, it is possible to choose a configuration $\mathbf{p}$ such that $G(\mathbf{p})$ is infinitesimally rigid in $\mathbb{E}^d$ and have an equilibrium stress with a stress matrix of maximal rank. But this does not mean that $G(\mathbf{p})$ itself is necessarily globally rigid in $\mathbb{E}^d$. For example, the bar framework of Figure (1a) has both matrices of maximal rank, but it is not globally rigid in the plane. See [9] for all kinds of examples along these lines along with a proof that coning in one higher dimension preserves generic global rigidity.

In order to describe specific configurations that are known to be globally rigid, we look more carefully at the proof of Theorem 17. We look at the sign of the stress of the edge that is being split and the way that the splitting node is chosen. Four possibilities are indicated in Figure 7 using the notation of Theorem 17 and Figure 6.

There are four other possibilities, also, where the roles of cables and struts have been reversed, but we wish to concentrate on the ones indicated by Figure 7.

Theorem 17 effectively says that when the additional node is added in the edge splitting operation, then an additional non-zero eigenvalue is created for the new tensegrity. It is possible to control the sign of the new eigenvalue.

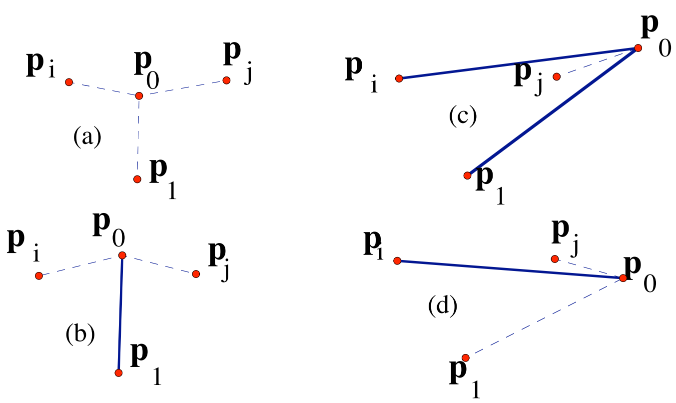

Figure 7: In all cases $\{i,j\}$ is the member being split. In cases (a) and (b) $\{i,j\}$ is a cable (i.e. $\omega_{ij} > 0$), and in cases (c) and (d) $\{i,j\}$ is a strut (i.e. $\omega_{ij} < 0$). The signs on the stresses after the splitting operation and perturbation are determined by the geometry. When $\mathbf{p}_0$ is not in the interval from $\mathbf{p}_i$ to $\mathbf{p}_j$, it does not matter which side it is on.

A simple three-node tensegrity is when $\mathbf{p}_i, \mathbf{p}_0, \mathbf{p}_j$ are order on a line, and it has an equilibrium stress where $\omega_{ij} < 0$, while $\omega_{i0} > 0$ and $\omega_{j0} > 0$. In other words, $\{i,j\}$ is a strut and $\{i,0\}$ and $\{0,j\}$ are cables. The stress matrix for this tensegrity is easily seen to be positive semi-definite with one positive eigenvlaue and two zero eigenvalues. So in the edge splitting operation, before the perturbation, one can choose the insertion of $\mathbf{p}_0$ and the two new members $\{i,0\}$ and $\{0,j\}$ replacing the previous $\{i,j\}$ as the addition of this small stress matrix the the original stress matrix in such a way that the cable stress say on $\{i,j\}$ cancels with the strut stress on the three node tensegrity. The effect is that the new eigenvalue will be positive, while the others will only be perturbed slightly. The same argument works when $\{i,j\}$ is a strut and $\mathbf{p}_0$ is on the line through $\mathbf{p}_i$ and $\mathbf{p}_j$, but not in the interval. In both cases we have the option of perturbing $\mathbf{p}_0$ in two of the cases of Figure 7.

Theorem 20. *If G is vertex 3-connected and generically redundantly rigid as a bar graph in $\mathbb{E}^2$, then there is a configuration $\mathbf{p}$ and a stress ω such that as tensegrity framework $G(\mathbf{p})$ is super stable.*

Proof. Start with the tensegrity of Figure 2, and apply the sequence of edge splittings by Jackson and Jordán as in Theorem 18. At each time use one of the possibilities of Figure 7. The tensegrity of Figure 2 is super stable, and conditions 2.) and 3.) of Theorem 3 are automatically preserved, and the argument above implies that 1.), the positive semi-definiteness, is preserved, also. So super stability is preserved. □

Note that if we insist that we only use options (a) and (d) of Figure 7, we can insure that the final super stable tensegrity has only two struts. Figure (3b) is an example. In that case we can make a more general statement.

First we need a Lemma.

Lemma 21. *Let $G(\mathbf{p})$ be tensegrity in the plane with n nodes, exactly two non-collinear struts and a proper equilibrium stress ω, non-zero on each member. Then the associated stress matrix Ω has maximal rank $n-3$. In other words, $\mathbf{p}$ is universal with respect to ω.*

Proof. Consider the universal configuration $\hat{\mathbf{p}}$ for the equilibrium stress ω. If the affine span of the nodes of $\hat{\mathbf{p}}$ are 3-dimensional (or higher), it is possible to find a plane that separates the two struts. Then by pushing toward to the plane we see that $\hat{\mathbf{p}}$ cannot be in equilibrium. Thus $\mathbf{p}$ is a universal configuration. □

Theorem 22. *Let $G(\mathbf{p})$ be tensegrity in the plane with n nodes, exactly two non-collinear struts, a proper equilibrium stress ω, non-zero on each member, and some cable not parallel to either strut. Then $G(\mathbf{p})$ is super stable.*

Proof. Construct another tensegrity $G_0(\mathbf{p})$ on the same vertices $\mathbf{p}$, with possibly more cables, but such that it has a proper equilibrium stress with stress matrix Ω_0 such that $\mathbf{p}$ is universal with respect to Ω_0. This can be obtained by adding positive semi-definite stresses of maximal rank defined on the two struts and one additional interior node. Figure 8 shows this. So Ω_0 is

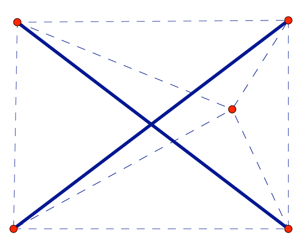

Figure 8: This is a super stable tensegrity with 5 nodes and 2 struts.

positive semi-definite with exactly 3 zero eigenvectors. For $0 \le t \le 1$ define $\Omega_t = (1-t)\Omega_0 + t\Omega$. So $\Omega_1 = \Omega$, and by Lemma 21, each Ω_t for $0 \le t \le 1$ has exactly 3 zero eigenvectors. Since the eigenvalues vary continuously with t the non-zero eigenvalues must remain positive. The condition on the cable directions insure that Condition 3.) for super stability holds. □

The analysis here is similar to what is done in [18], but that paper is not concerned with the global rigidity of tensegrities, only the infinitesimal rigidity. Figure 7 and Figure 2 in [18] are very similar, except (C) and (D) in their Figure 2 create a negative eigenvalue for the associated stress matrix.

It would be interesting to be more explicit as to what the two struts are. The proof of Theorem 22 chooses the struts depending on the edge-splitting sequence of Jackson-Jordán in Theorem 18. But, nevertheless, it is not possible to choose the two disjoint struts at will. For example, if the struts as chosen as in (Figure 9a), there is no realization in $\mathbb{E}^2$, whose affine span is 2-dimensional and is in equilibrium with a strict proper stress, since there is no position for $\mathbf{p}_5$ in (Figure 9b). This the same as Figure (14G) in [18]. See also [20] for an analysis of the sign patterns of rigid tensegrities.

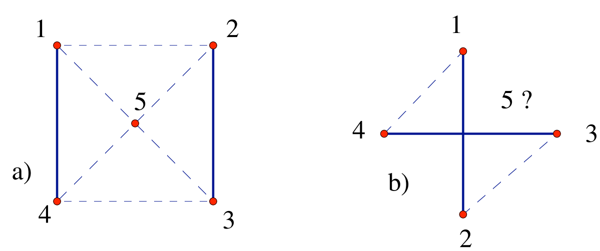

Figure 9: Figure a) represents an abstract choice of struts and cables for a tensegrity, and Figure b) shows that since the two struts must cross when there is an equilibrium stress, there is no choice for the position of $\mathbf{p}_5$.

Question 3. *Let G be a vertex 3-connected and generically redundantly rigid bar graph in $\mathbb{E}^2$, is there a reasonable characterization of which choices of a and b as two disjoint struts will yield a strict proper equilibrium stress as in Theorem 22?*

10 Special tensegrities

10.1 Strut triangles

If three nodes are pinned in the plane, and positive stresses ω are chosen for a graph G, then there is a unique configuration $\mathbf{p}$ such that $\mathbf{p}$ is in equilibrium with respect to ω, except for the pinned points. But then it is possible to place struts between each pair of pinned vertices and assign negative stresses on those struts such that the whole tensegrity is in equilibrium. This is a version of Tutte's paper "How to draw a graph" in [24] and the "spider webs" of [4]. If each node is connected to all three originally pinned nodes, the resulting stress matrix will be positive semi-definite, maximal rank, and rigid. Thus it will be super stable. Note that G does not have to be infinitesimally rigid in the plane, as in Figure 10. But if G is generically redundantly rigid, there is the following question, which is similar to Question 3.

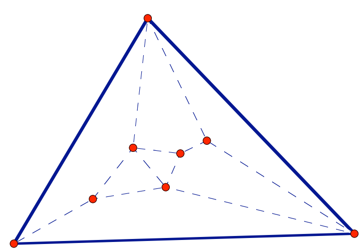

Figure 10: A super stable non-infinitesimally rigid tensegrity. The two triangles in the center have an infinitesimal rotational motion about a point inside.

Question 4. *Let G be a vertex 3-connected and generically redundantly rigid bar graph in $\mathbb{E}^2$, and let a, b and c three edges forming a triangle. Is it possible to arrange it so that a, b and c are the only struts after the edge splitting operations of Theorem 22?*

For example, can we start with a triangle and one vertex inside, and only perform the edge-splitting such that the edges of the original triangle are never split? If G has a configuration as a tensegrity that is infinitesimally rigid, it is not necessarily true that all the super stable configurations with positive stresses inside the triangle are infinitesimally rigid, as with Figure 10. But for this graph there are infinitesimally rigid configurations, which can be found by using edge splittings on the interior edges.

Tibor Jordan has kindly pointed out that when the graph G is a 3-connected generic rigidity circuit, i.e. $m = 2n - 2$, where n is the number of nodes and m is the number of members, and G is a 3-connected generically redundantly rigid graph, then Theorem 4 in [3] implies that the answer to Question 4 is affirmative. This is because their Theorem 4 says that any 3-connected generic rigidity circuit, with $n \geq 5$, has at least 2 disconnected vertices of degree 3, where the inverse operation of deleting the vertex along with its incident edges and joining some pair of the incident vertices by a bar, can be performed. So any given triangle can be avoided, when the edge splitting operation is performed.

10.2 Strut Hamiltonian cycles

There are other geometric classes of super stable tensegrities. In [4] it is shown that any cable polygon in the plane with struts as diagonals, such that it has a proper equilibrium stress, is super stable. This might suggest that if a graph G is vertex 3-connected, generically redundantly rigid in the plane, and has a Hamiltonian cycle, then it can be obtained by starting with Figure 2, and perform only operations of type (b) in Figure 7. But this is not always possible as seen in Figure 11.

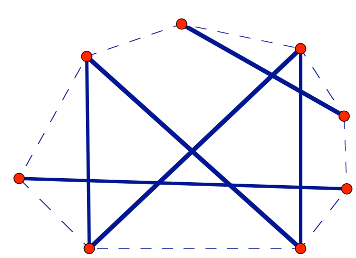

Figure 11: The underlying bar graph of this tensegrity is vertex 3-connected and redundantly rigid in the plane. But it is not possible to achieve this configuration by operations of type (b) in Figure 7 since the four nodes of degree three prevent it.

On the other hand, I believe that the following is true, possibly by using energy techniques:

Conjecture 1. *If G is a vertex 3-connected graph with a Hamiltonian cycle, then there is a configuration $\mathbf{p}$ in the plane, with the edges of the Hamiltonian cycle as cables, all other members struts, and a proper equilibrium stress.*

Note that this does not assume that G is generically redundantly rigid in the plane. Figure 12 shows this sort of example. This example is globally rigid, but not at a generic configuration.

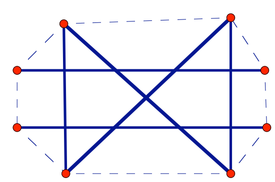

Figure 12: A configuration with a stress that makes the tensegrity universally globally rigid, but it is not generically globally rigid in the plane, even as a bar framework.

In the spirit of looking for configurations for tensegrities that have strong rigidity proproperties, it is interesting to note that Roth polygons, Cauchy polygons and others (as shown in Figure 13) and defined in [4]) have the property that they are always superstable and infinitesimally rigid for convex configurations.

Question 5. *What are some other examples of tensegrity polygons with only cables on the edges with all the other members struts, where the tensegrities are superstable and infinitesimally rigid in the plane?*

10.3 Higher dimensional tensegrities

The Tutte realization of a graph with a simplex as a subgraph is still possible. And for such configurations the stresses on the internal cables can be chosen at will. Unfortunately, for higher dimensions, we cannot always expect the Tutte/spider web realization to be infinitesimally rigid. This is a result in [10]

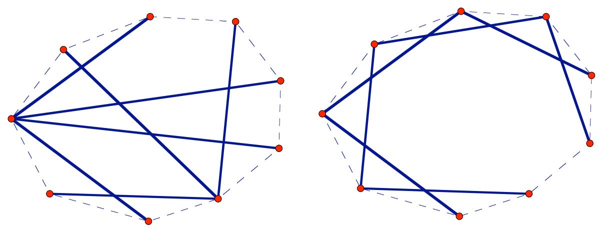

Figure 13: A Roth polygon and a Cauchy polygon. The Roth polygon is a convex cable polygon, there two nodes that are connected by daigonal struts to all the other nodes. For a Cauchy polygon each node is connected by a strut to two nodes two steps away, except for two adjacent struts in the sequence.

Theorem 23. *There is a graph G that is vertex $(d+1)$-connected, generically redundantly rigid in $\mathbb{E}^d$, for $d \geq 5$ and has a d-simplex (i.e. K_{d+1}) as a subgraph, but it is not generically globally rigid?*

Note that since G is generically globally rigid as a bar framework in $\mathbb{E}^d$ if there were an infinitesimally rigid realization in $\mathbb{E}^d$ it would be automatically super stable and generically globally rigid. But since it is not generically globally rigid, it cannot have been infinitesimally rigid in its spider web realization.

In $\mathbb{E}^3$, the only vertex 4-connected, generically redundantly rigid graph in $\mathbb{E}^3$ that is not generically globally rigid, that I know of, is the complete bipartite graph $K_{5,5}$, and it does not contain any 3-simplex, i.e. K_4 as a subgraph.

Note that by gluing graphs along the simplex, we get the following.

Corollary 24. *For $d \geq 5$ are there an infinite number of isomorphism classes of graphs G that are vertex $(d+1)$-connected, generically redundantly rigid in $\mathbb{E}^d$ but not generically globally rigid in $\mathbb{E}^d$?*

11 Vertex splitting

A very useful operation that was described by Walter Whiteley in [27] takes a graph that is generically rigid and adds one new vertex with appropriate rearranging and the addition of new edges to get a new generically rigid bar framework in $\mathbb{E}^d$. This is described next for a graph G, with vertices labeled $(1, 2, \ldots, n)$. We will split the vertex labeled 1. Assume that $\{1,2\}, \{1,3\}, \ldots, \{1,d\}$ and $\{1, d+1\}, \ldots, \{1, d+k_1\}$, and $\{1, d+k_1+1\}, \ldots, \{1, d+k_1+k_2\}$ are all the edges adjacent to the vertex 1. Remove the edges $\{1, d+1\}, \ldots, \{1, d+k_1\}$, add a new vertex labeled 0, add the edges $\{0,1\}, \{0,2\}, \{0,3\}, \ldots, \{0,d\}$ and $\{0, d+1\}, \ldots, \{0, d+k_1\}$. Call this new graph the d-dimensional vertex split of G.

Another way of thinking of this is to remove the vertex 1 and its adjacent edges, write its neighbors as the union of two sets A and B with exactly $d-1$ vertices in $A \cap B$, and join 0 to A and 1 to B. Note that k_1 or k_2 can be 0. Note also that one new vertex is added, and a total of d additional edges added.

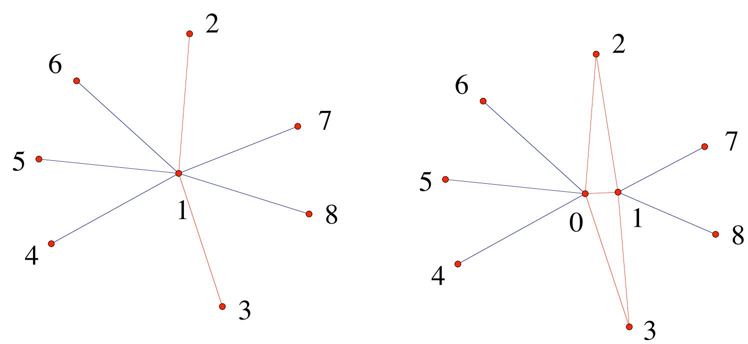

Figure 14: This shows the vertex-splitting operation for $d = 3$, $k_1 = 3$, and $k_2 = 2$.

The point of this operation is the following in [27].

Theorem 25. *If a graph G is generically rigid in $\mathbb{E}^d$, then so is G_1 obtained from G by the d-dimensional splitting of any vertex.*

The following was conjectured by Walter Whiteley.

Conjecture 2. *If a graph G is generically globally rigid in $\mathbb{E}^d$, then so is G_1 obtained from G by the d-dimensional splitting of any vertex, where $k_1 \geq 1$ and $k_2 \geq 1$.*

Conjecture 2 has already been proved in the case $d = 2$ in [19] using purely combinatorial methods. In order to understand the results here we need to look at the analysis in [27].

Consider the rigidity matrix $R(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ for the bar graph $G(\mathbf{p}_1, \ldots, \mathbf{p}_n)$, where the vertex 1 is to be split as in the notation described earlier. We would like to place $\mathbf{p}_0 = \mathbf{p}_1$, but the edge $\{0, 1\}$ would have 0 length, and this makes stress calculations awkward. But the rank of a rigidity matrix is not changed by multiplying any row by a non-zero scalar. So we define a matrix that would be the rigidity matrix for the graph G_1 with $\mathbf{p}_0 = \mathbf{p}_1$, except that for the row corresponding to the edge $\{0, 1\}$, instead of being the 0 row, the $\mathbf{p}_1 - \mathbf{p}_0$ and $\mathbf{p}_0 - \mathbf{p}_1$ entries are replaced by a non-zero vector $\mathbf{d}_{10}$ and $\mathbf{d}_{01} = -\mathbf{d}_{10}$, respectively. Call this matrix $R_1(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n)$, which is a function of not just the vertices $(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n)$ but also $\mathbf{d}_{01}$, which we call the *residual direction*.

Lemma 26. *If the vectors $(\mathbf{d}_{01}, \mathbf{p}_2 - \mathbf{p}_1, \ldots, \mathbf{p}_d - \mathbf{p}_1)$ are independent (i.e. a basis), then dimension of the co-kernel of $R(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ and $R_1(\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n)$ are the same.*

Proof. Let ω be any equilibrium stress for $G(\mathbf{p}_1, \ldots, \mathbf{p}_n)$, and define

$$\mathbf{v}_0 = \sum_{i=d+1}^{d+k_1} \omega_{1i}(\mathbf{p}_i - \mathbf{p}_1) \text{ and } \mathbf{v}_1 = \sum_{i=d+k_1+1}^{d+k_1+k_2} \omega_{1i}(\mathbf{p}_i - \mathbf{p}_1),$$

while the equilibrium condition at $\mathbf{p}_1$ implies that

$$\mathbf{v}_0 + \sum_{i=2}^{d} \omega_{1i}(\mathbf{p}_i - \mathbf{p}_1) + \mathbf{v}_1 = \mathbf{0}. \tag{8}$$

Since $(\mathbf{d}_{01}, \mathbf{p}_2 - \mathbf{p}_1, \ldots, \mathbf{p}_d - \mathbf{p}_1)$ is a basis for $\mathbb{E}^d$, there are unique scalars $\hat{\omega}_{12}, \ldots, \hat{\omega}_{1d}$ and ω_{01} such that

$$\sum_{i=2}^{d} \hat{\omega}_{1i}(\mathbf{p}_i - \mathbf{p}_1) + \omega_{01}\mathbf{d}_{01} = -\mathbf{v}_1,$$

and similarly there are unique scalars $\hat{\omega}_{02}, \ldots, \hat{\omega}_{0d}$ and ω_{10} such that

$$\sum_{i=2}^{d} \hat{\omega}_{0i}(\mathbf{p}_i - \mathbf{p}_0) + \omega_{10}\mathbf{d}_{10} = -\mathbf{v}_0.$$

Adding these last two equations and using (8), we get

$$\sum_{i=2}^{d}(\hat{\omega}_{1i}+\hat{\omega}_{0i})(\mathbf{p}_i-\mathbf{p}_1)+(\omega_{10}-\omega_{01})\mathbf{d}_{10}=\sum_{i=2}^{d}\omega_{1i}(\mathbf{p}_i-\mathbf{p}_1).$$

Thus $\hat{\omega}_{1i}+\hat{\omega}_{0i}=\omega_{1i}$ for $i=2,\dots,d$, and $\omega_{10}=\omega_{01}$. Thus we have an isomorphism from the co-kernel of equilibrium stresses of $G(\mathbf{p}_1,\dots,\mathbf{p}_n)$ to the co-kernel of $R_1(\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n)$ where ω_{ij} is sent to $\hat{\omega}_{ij}$ when i or j is 1, while ω_{01} is determined as above and the other stresses are not changed. $\square$

Let us call the ω_{01}, as defined above, the *bridging stress* for $G(\mathbf{p}_1,\dots,\mathbf{p}_n)$ for the choices for the vertex splitting giving G_1.

One important observation is that the rank of $R_1(\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n)$ is not changed if the rank of $R(\mathbf{p}_1,\dots,\mathbf{p}_n)$ is maximal and one perturbs $\mathbf{p}_0$ to be different from $\mathbf{p}_1$, while $(\mathbf{p}_0-\mathbf{p}_1)/|\mathbf{p}_0-\mathbf{p}_1|$ is close to $\mathbf{d}_{01}/|\mathbf{d}_{01}|$. Also, similar to the situation with edge splitting, the space of stresses themselves will vary continuously as $\mathbf{p}_0$ is moved nearby $\mathbf{p}_1$ making sure that $\mathbf{p}_0-\mathbf{p}_1$ is not included in any linear combination of the $d-2$ neighbors of the original $\mathbf{p}_1$. For example, $\mathbf{p}_0$ could be restricted to lie in an appropriate open cone centered at $\mathbf{p}_1$.

We can also get geometric information for super stable infinitesimally rigid tensegrites.

Theorem 27. *Suppose a tensegrity $G(\mathbf{p}_1,\dots,\mathbf{p}_n)$ is super stable at a generic configuration $(\mathbf{p}_1,\dots,\mathbf{p}_n)$ in $\mathbb{E}^d$, and G_1 is obtained from G by d-dimensional splitting of the vertex $\mathbf{p}_1$, where $k_1\ge 1$ and $k_2\ge 1$ and a bridging stress is non-zero. Then there is generic choice for $\mathbf{p}_0$ such that $G_1(\mathbf{p}_0,\mathbf{p}_1,\dots,\mathbf{p}_n)$ is also super stable.*

Figure 15 shows how Theorem 27 works. Figure (15a) is super stable and is split at vertex 1. When $\mathbf{p}_0$ is chosen as in Figure (15b) it is super stable, but in the case of Figure (15c) the stress matrix is not positive semi-definite and the tensegrity is not globally rigid.

What is used here is that as $\mathbf{p}_0$ approaches $\mathbf{p}_1$, the stress ω_{01} goes to infinity. The following Lemma concerning symmetric matrices and quadratic forms helps.

Lemma 28. *Let Ω_t for $0\le t<1$ be a one parameter family of symmetric n-by-n matrices, where each entry is a continuous function of t. Let $U\oplus V$ be a fixed orthogonal decomposition of $\mathbb{R}^n$, where U and V are linear subspaces*

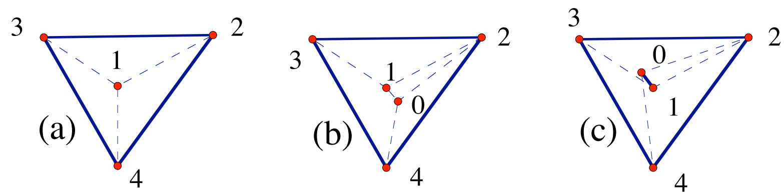

Figure 15: Tensegrity (a) is split at vertex 1 and in both cases (b) and (c) the resulting stress matrix is of rank 2, the maximum. Although tensegrity (c) is not globally rigid in the plane, as a bar framework it is globally rigid in the plane.

of dimension $n - 1$ *and* 1, *respectively. Assume that the quadratic form* Q_t *associated with* Ω_t, *when restricted to* U, *converges to a finite quadratic form* Q_1 *as* $t \to 1$, *and trace*$(\Omega_t) \to \infty$ *as* $t \to 1$. *Then the eigenvalues of* Ω_t *converge to the eigenvalues of* Q_1, *with the exception of one eigenvalue whose limit is* ∞. *(There is a similar statement for* $-\infty$ *replacing* ∞.*)*

Proof. It is clear that as $t \to 1$, at least one of the eigenvalues of Ω_t go to ∞. Any eigenvector associated to that increasingly large eigenvalue, must converge to a vector in V, since otherwise it would have a component in U and Q_t would not converge to Q_1. Thus all the other eigenvectors must converge to vectors in U. $\square$

Proof of Theorem 27. From Lemma 26 the rank of $R(\mathbf{p}_1, \dots, \mathbf{p}_n)$ and $R_1(\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n)$ are the same and maximal since $G(\mathbf{p}_1, \dots, \mathbf{p}_n)$ is assumed to have maximal rank. So when $\mathbf{p}_0$ is not equal to $\mathbf{p}_1$ the ordinary rigidity matrix has maximal rank as well, and the space of equilibrium stresses varies continuously as long as $\mathbf{p}_0$ stays in a cone centered at $\mathbf{p}_1$ mentioned above. Choose a path so that $\mathbf{p}_0$ converges to $\mathbf{p}_1$ in this cone, and let Ω_t as $t \to 1$ be a continuous choice of a stress matrix for these configurations, so that the entries Ω_t each converge to corresponding entry of Ω, the n-by-n positive semi-definite stress matrix for $G(\mathbf{p}_1, \dots, \mathbf{p}_n)$ given by the super stability hypothesis. The following shows a portion of the stress matrix.

$$\Omega_t = \begin{pmatrix} \omega_{01} + \text{finite} & -\omega_{01} & -\omega_{02} & \cdots & -\omega_{0d} & \cdots \\ -\omega_{01} & \omega_{01} + \text{finite} & -\omega_{12} & \cdots & -\omega_{1d} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\omega_{0d} & -\omega_{1d} & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Define the following matrix n-by-$(n-1)$ matrix

$$X = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Then $X^T \Omega_t X$ converges to the matrix Ω. So the eigenvalues of $X^T \Omega_t X$ converge to the eigenvalues of Ω. Since we have assumed that bridging stress is non-zero, the stress ω_{01} in Ω_t must converge to ∞ as $t \to 1$. By choosing $\mathbf{p}_0$ on the appropriate side of the cone for taking the limit, we can be sure that $\omega_{01} > 0$. Then Lemma 28 applies and we find that for t close enough to 1, Ω_t is positive semi-definite of maximal rank. □

In the argument above for the proof of Theorem 27, if we only have control over the rank of Ω, we can still recover a weakened form of Conjecture 2.

Theorem 29. *Suppose a tensegrity $G(\mathbf{p}_1, \dots, \mathbf{p}_n)$ is globally rigid at a generic configuration $(\mathbf{p}_1, \dots, \mathbf{p}_n)$ in $\mathbb{E}^d$, and G_1 is obtained from G by d-dimensional splitting of the vertex $\mathbf{p}_1$, where $k_1 \geq 1$ and $k_2 \geq 1$ and at a generic configuration the stress $\omega_{01} \neq 0$. Then there is generic choice for $\mathbf{p}_0$ such that $G_1(\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n)$ is generically rigid.*

Figure 16 shows one of the difficulties of Theorem 29 and Theorem 27. As vertex 1 is split in the framework (16a), the new $\{0, 1\}$ member in the split framework (16b) has a 0 stress, and the dimension of the kernel of the corresponding stress matrix increases by one. However, the framework in Figure (16a) is not generically globally rigid in $\mathbb{E}^3$.

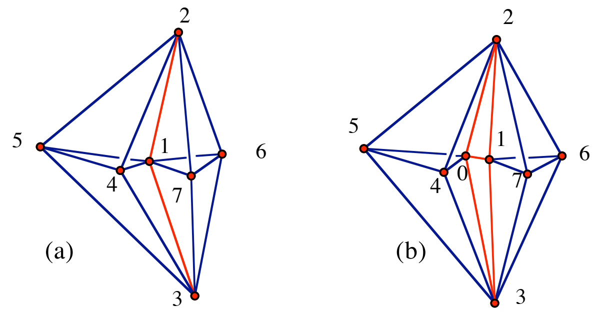

Figure 16: A vertex splitting in dimension 3, with the labeling convention as in Figure 14.

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